**Total No. of Questions : 8]** **[Total No. of Pages : 3**

**P804**

# [3621] - 102
# M.A. / M.Sc.
# MATHEMATICS
## MT-502: Advanced Calculus
## (Old & New)

***Time : 3 Hours]*** ***[Max. Marks :80***

***Instructions to the candidates:***

1) *Attempt any five questions.*
2) *Figures to the right indicates full marks.*

***Q1)*** a) Define Continuity of a vector field. Prove that the dot product of two continuous vector fields is also continuous. **[5]**

b) Define directional derivative of a scalar field. Show that the existence of directional derivative at point need not imply the continuity of the function. **[6]**

c) State and prove chain rule for differentiability of a scalar fields. **[5]**

***Q2)*** a) State only implicit function theorem. **[2]**

Let $\vec{f} = (f_1, f_2)$ be a mapping from $R^2$ to $R^2$ given by $f_1(x, y) = e^x \cos y$ and $f_2(x, y) = e^x \sin y$.

Show that Jacobian of $\vec{f}$ is not zero in $R^2$. Let $\vec{a} = (0, \pi/3), \vec{b} = \vec{f}(\vec{a})$ and let $\vec{g}$ be the continuous inverse of $\vec{f}$ in the neighbourhood of $\vec{b}$ such that $\vec{g}(\vec{b}) = \vec{a}$. Find explicit formula for $\vec{g}$ and compute $\vec{f}'(\vec{a})$ and $\vec{g}'(\vec{b})$ and verify $\vec{g}'(\vec{b}) = \{\vec{f}'(\vec{g}(\vec{b}))\}^{-1}$. **[6]**

**OR**

Let $f : S \subset R^n \to R$ be a scalar field defined on S in $R^n$. Assume that the partial derivatives $D_1 f, D_2 f, \ldots, D_n f$ exist in some n-ball $B(\vec{a}) \subset S$ and are bounded in $B(\vec{a})$. Then prove that $f$ is continuous at $\vec{a}$. **[8]**

b) The substitution $u = \frac{x-y}{2}, v = \frac{x+y}{2}$ changes $f(u, v)$ into $F(x, y)$. Express the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ in terms of the partial derivatives $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$. **[4]**

c) Assume that $f'(\vec{x},\vec{y})=0$ for all $\vec{x}\in B(\vec{a})$ and for all $\vec{y}$. Show that $f$ is constant on $B(\vec{a})$. **[4]**

***Q3)*** a) Define line integral of a vector field. State only the basic properties of the line integral.

State and prove the behaviour of the line integral under a change of a parameters. **[8]**

b) Evaluate $\int_C (x^2-2xy)\,dx+(y^2-2xy)\,dy$, where C is a path from (-2, 4) to (1, 1) along the parabola $y = x^2$. **[4]**

c) Find the amount of work done by the force $\vec{f}(x,y)=(x^2-y^2)\vec{i}+2xy\vec{j}$ in moving a particle in counter clockwise direction once around the square bounded by the coordinate axes and the lines $x = a$ and $y = a$, $a > 0$. **[4]**

***Q4)*** a) Let $\vec{f}$ be a continuous vector field on an open connected set $S\subset R^n$. If the line integral of $\vec{f}$ is independent of the path in S, prove that $\vec{f}$ is a gradient of some potential function on S. What about the converse? Justify your Answer. **[10]**

b) Is the function $\vec{f}(x,y)=x\vec{i}+xy\vec{j}$ gradient of some potential function? Compute $\int_C \vec{f}\cdot d\vec{\alpha}$, where $C:\vec{\alpha}(t)=a\cos t\,\vec{i}+a\sin t\,\vec{j},\ 0\le t\le 2\pi$. **[6]**

***Q5)*** a) Let $f$ be bounded on a rectangle Q in $R^2$. Show that upper integral $\bar{I}(f)$ and lower integral $\underline{I}(f)$ exist. Next prove that $f$ is integrable over Q if and only if $\bar{I}(f)=\underline{I}(f)$. **[6]**

b) Let $f$ be defined on a rectangle Q = [0, 1] × [0, 1] as follows.

$$f(x,y)=\begin{cases}1-x-y & \text{if } x+y\le 1\\ 0 & \text{otherwise}\end{cases}$$

Make a sketch of the ordinate set of $f$ over Q and compute volume of this ordinate set by double integration. **[6]**

c) Evaluate $\iint_S (x^2-y^2)\,dx\,dy$, where S is bounded by the curve $y = \sin x$ and $x \in [0, \pi]$. **[4]**

***Q6)*** a) State the formula for change of variables in double integrals. Prove this formula for particular case when the region of integration is rectangle and the function with constant value 1. **[10]**

b) Evaluate using Green's theorem the line integral $\int_C Pdx + Qdy$, around the boundary of the square of a side $2a$ determined by the inequalities $|x| \leq a$ and $|y| \leq a$, where $P(x, y) = xe^{-y^2}$ and $Q(x, y) = -x^2 ye^{-y^2} + \frac{1}{x^2 + y^2}$. **[6]**

***Q7)*** a) Define the fundamental vector product on a surface. Find $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$, where

$\vec{r}(u,v) = a \cos u \cos v \ \vec{i} + a \sin u \cos v \ \vec{j} + a \sin v \ \vec{k}$

$(u,v) \in T = [0, 2\pi] \times [0, \frac{\pi}{2}]$. Discuss the singular points of this surface.

**[6]**

b) If S denote the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$, and $\vec{F}(x, y, z) = x\vec{i} + y\vec{j}$. Let $\vec{n}$ be the unit outward normal of S. Compute $\iint_S \vec{F} \cdot \vec{n} ds$, where S has vector representation. **[6]**

c) Evaluate $\iint_S \sqrt{a^2 - x^2 - y^2} dxdy$, where S is region in the first quadrant of the circular disc $x^2 + y^2 \leq a^2$. **[4]**

***Q8)*** a) State and prove Stokes theorem. **[10]**

b) Let $\vec{f}(x, y) = \frac{-y}{x^2 + y^2}\vec{i} + \frac{x}{x^2 + y^2}\vec{j}, (x, y) \in R^2 \setminus \{(0,0)\}$. Show that curl and divergence of $\vec{F}$ are zero on $R^2 \setminus \{(0, 0)\}$. **[6]**

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**Total No. of Questions : 8]** **[Total No. of Pages : 2**

**P806**

# [3621] - 104
# M.A. / M.Sc.
# MATHEMATICS
## MT-504 : Number Theory
## (Old & New)

***Time : 3 Hours]*** ***[Max. Marks :80***

***Instructions to the candidates:***

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

***Q1)*** a) Show that every integer greater than one can be written as product of Primes. Further, show that this factorization is unique a part from the order in which primes occur. **[8]**

b) Find greatest common divisor of 696 and 372 and express it as their linear combination. **[4]**

c) Show that there are arbitrarily large gaps in the sequence of primes. **[4]**

***Q2)*** a) Let $m$ and $a$ be positive integers such that $(a, m) = 1$. Prove that $a^{Q(m)} \equiv 1(\text{mod } m)$. **[6]**

b) Prove that $\frac{1}{5}n^5 + \frac{1}{3}n^3 + \frac{7}{15}n$ is an integer for every $n$. **[4]**

c) Find the smallest positive integer giving remainders 1,2,3,4 and 5 when divided by 3, 5, 7, 9 and 11. **[6]**

***Q3)*** a) Let $p$ be an odd prime and $(a, 2p) = 1$. Prove that $\left(\frac{a}{p}\right) = (-1)^t$, where

$$t = \sum_{j=1}^{(p-1)/2} \left[\frac{ja}{p}\right] \text{and} \left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}.$$ **[7]**

b) Prove that there are infinitely many primes of the form $6n + 1$. **[5]**

c) Let $p$ and $q$ be primes such that $q = p + 2$. Prove that there is an integer $a$ such that $p| \, a^2 - q$ if and only if there is an integer $b$ such that $q| \, b^2 - p$. **[4]**

**P.T.O.**

*Q4)* a) Let $p$ be a prime. Prove that the largest exponent $e$ such that $p^e \mid n!$ is

$$e = \sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right].$$ **[6]**

b) Let $x, y$ be real numbers. Prove that $[x] + [y] \le [x + y] \le [x] + [y] + 1$. **[4]**

c) Define $w(n)$. Prove that for every positive integer $n$, $\sum_{d|n} \mu(d)\sigma(d) = (-1)^{W(n)}$.

Also evaluate $\sum_{d|n} \mu(d)\sigma(d)$. **[6]**

*Q5)* a) Determine all primitive positive solutions of the equation $x^2 + y^2 = z^2$. **[6]**

b) Find all integers $x$ and $y$ such that $147x + 258y = 369$. **[5]**

c) Let $p$ be a prime. If $a \equiv b \pmod{p^n}$ then prove that $a^p \equiv b^p \pmod{p^{n+1}}$. **[5]**

*Q6)* a) Let $\alpha$ be an algebraic number of degree $n$. prove that every number in $Q(\alpha)$ can be written uniquely in the form $a_0 + a_1\alpha + \ldots\ldots + a_{n-1}\alpha^{n-1}$, where $a_i$ are rational numbers. **[6]**

b) Prove that $11 + 2\sqrt{6}$ is *a* prime in $Q(\sqrt{6})$ but 3 is not a prime in $Q(\sqrt{6})$. **[4]**

c) Let $m$ be an integer such that $m \equiv 5 \pmod 8$. If $Q(\sqrt{m})$ has the unique factorization property then prove that 2 is prime in $Q(\sqrt{m})$. **[6]**

*Q7)* a) Prove that the quadratic field $Q(\sqrt{-3})$ is Euclidean. **[7]**

b) Find a polynomial of degree 4 satisfied by $1 + \sqrt{2} + \sqrt[4]{2}$. Is it a minimal polynomial? **[6]**

c) Prove that $\sqrt{3}+1$ and $\sqrt{3}-1$ are associates in the ring of integers of $Q(\sqrt{3})$. **[3]**

*Q8)* a) Prove that $ax \equiv b \pmod m$ has a solution if and only if $(a, m) | b$. Further, if $(a, m) \mid b$ then prove that $ax \equiv b \pmod m$ has exactly $(a, m)$ solutions incomplete residue system modulo $m$. **[6]**

b) Prove that $18! \equiv -1 \pmod{437}$. **[4]**

c) Let $n$ be a positive integer. prove that $q = 4^n + 1$ is a prime if and only if $3^{(q-1)/2} \equiv -1 \pmod q$. **[6]**

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**P808**

# [3621] - 201
# M.A. / M.Sc.
# MATHEMATICS
## MT-601 : General Topology
## (New)

*Time : 3 Hours]* *[Max. Marks :80*

***Instructions to the candidates:***

1) ***Answer any five questions.***
2) ***Figures to the right indicate full marks.***

***Q1)*** a) Let X be a topological space. Suppose that C is a collection of open sets of X such that for each open set U of X and each $x$ in U, there is an element $c$ of C such that $x \in c \subseteq U$. Show that C is a basis for the topology of X. **[6]**

b) i) Is union of two topologies on a set X a topology on X ? Justify your answer.

ii) Show that the topologies of $R_e$ and $R_k$ are not comparable. **[5]**

c) Consider the set Y = [–1, 1] as a subspace of R. Which of the following sets are open in Y? Which are open in R? **[5]**

$$A = \{x / \frac{1}{2} < |x| \leq 1\}$$

$$B = \{x / 0 < |x| < 1 \text{ and } \frac{1}{x} \notin Z_+\}.$$

***Q2)*** a) Let X be an ordered set in the order topology and let Y be a subset of X that is convex in X. Show that the order topology on Y is the same as the topology Y inherits as a subspace of X. **[6]**

b) i) What are closed sets in the finite complement topology on R?

ii) Find the set of limit points of the subset Z of R under usual topology. **[5]**

c) i) Show that every order topology is Hausdorff.

ii) Give an example of a topological space which is not a Hausdorff space. **[5]**

*Q3)* a) Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation $f(a) = (f_\alpha(a))_{\alpha \in J}$, where $f_\alpha : A \to X_\alpha$ for each $\alpha$. Let $\Pi X_\alpha$ have the product topology. Show that the function $f$ is continuous if and only if each function $f_\alpha$ is continuous. **[6]**

b) Let X be a metric space with metric d. Show that $d : X \times X \to R$ is continuous. **[5]**

c) Show that the subspace $(a, b)$ of R is homomorphic to (0,1). Further, show that the subspace [0, 1] of R is not homomorphic to the subspace $S^1$ of $R^2$. **[5]**

*Q4)* a) Show that $R^w$ in the box topology is not metrizable. **[6]**

b) Let $f : S^1 \to R$ be a continuous map. Show that there exists a point $x$ of $S^1$ such that $f(x) = f(-x)$. **[5]**

c) Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that pof equals the identity map of Y, then p is a quotient map. **[5]**

*Q5)* a) Prove that a finite cartesian product of connected spaces is connected. **[6]**

b) Assume that R is uncountable. Show that if A is a countable subset of $R^2$, then $R^2 - A$ is path connected. **[5]**

c) What are components and path components of $R_l$? What are continuous maps $f : R \to R_l$ ? **[5]**

*Q6)* a) Let X be a non-empty compact Hausdorff space. If X has no isolated points, then show that X is uncountable. **[6]**

b) Show that compactness implies limit point compactness, but not conversely. **[5]**

c) i) Show that the image of a compact space under a continuous map is compact.

ii) What is the smallest compact space containing the set

$$x = \left\{ \tfrac{1}{n} \mid n \in Z_+ \right\}?$$ **[5]**

*Q7)* a) Show that the space $R_k$ is Hausdorff but not regular. **[6]**

b) Show that one point compactification of R is homomorphic with the circle $S^1$. **[5]**

c) Show that every compact metrizable space X has a countable basis. **[5]**

*Q8)* a) Show that every compact Hausdorff space is normal. **[6]**

b) Show that $X \times Y$ is a regular space if and only if both X and Y are regular spaces. **[5]**

c) Show that a connected normal space having more than one point is uncountable. **[5]**

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**P808**

# [3621] - 201
# M.A. / M.Sc.
# MATHEMATICS
## MT-601 : Real Analysis - II
## (2005 Pattern) (Old Course)

*Time : 3 Hours]*  *[Max. Marks :80*

***Instructions to the candidates:***

1) ***Answer any five questions.***
2) ***Figures to the right indicates full marks.***

***Q1)*** a) Prove that BV[$a$, $b$] is complete under $\|f\|_{BV} = |f(a)| + V_a^b f$ **[8]**

b) With usual notation prove that :

$$\|f_1 f_2\|_{BV} \le \|f_1\|_{BV} \, \|f_2\|_{BV}$$ **[6]**

c) State Helly's first theorem. **[2]**

***Q2)*** a) Let $\alpha : [a, b] \rightarrow R$ be increasing. Then prove that a bounded function $f : [a, b] \rightarrow R$ is in $R_\alpha [a, b]$ iff given $\varepsilon > 0$ these exist a partition p of $[a, b]$ such that $U(f, p) - L(f, p) < \varepsilon$. **[8]**

b) Let $f, g \in R_\alpha [a, b]$ and let $C \in R$ then prove that $Cf \in R_\alpha [a, b]$ and

$$\int_a^b c\, f\, d\alpha = c \int_a^b f\, d\alpha \,.$$ **[6]**

c) State Riesz Representation theorem. **[2]**

***Q3)*** a) Let $E \subset R$. Define outer measure M*(E). Show that **[8]**

i) $m^*(E) = 0$ if E is countable.

ii) $m^*(E) \le m^*(F)$, where $E \subseteq F$.

b) Prove that E is measurable iff $E \cap (a, b)$ is measurable for every bounded open interval $(a, b)$. **[8]**

***Q4)*** a) Show that the collection of all measurable subsets (M) forms an algebra. **[6]**

b) Let $\{E_n\}$ be a collection of measurable sets with $E_n \supset E_{n+1}$ and if some $E_k$ has finite measure then prove that $m(\bigcap_{n=1}^{\infty} E_n) \lim_{n \to \infty} m(E_n)$. **[5]**

**P.T.O.**

c) Let $f$ : B $\rightarrow$ R and B is measurable set then prove that $f$ is measurable function iff $f^{-1}$ (U) is measurable for every open set U $\subseteq$ R. **[5]**

***Q5)*** a) State and prove monotone convergence theorem. **[8]**

b) Define simple function and if $\phi$ and $\Psi$ are integrable functions and $\alpha, \beta \in$ R then prove that $\int(\alpha\phi + \beta\psi) = \alpha \int\phi + \beta \int\psi$. **[6]**

c) State Fatou's Lemma. **[2]**

***Q6)*** a) Give an example of a improper Riemann Integrable function which is not Lebesgue integrable. **[8]**

b) State and prove Lebesgue dominated convergence theorem. **[8]**

***Q7)*** a) Define convergence in measure. Show that convergence in measure doesn't imply convergence in $L_1$. **[8]**

b) State and prove Holder's inequality. **[8]**

***Q8)*** a) If $f$ : $[a, b] \rightarrow$ R is a bounded Riemann integrable function then prove that $f$ is also Lebesgue integral on $[a, b]$ and the two integrals agree ie.

$$(R)\int_a^b f(x)\, dx = (L)\int_a^b f.$$ **[8]**

b) Let $f$ be non-negative and measurable then prove that $\int f = 0$ iff $f = 0$ almost everywhere. **[6]**

c) State Egorov's theorem. **[2]**

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**Total No. of Questions : 8]** **[Total No. of Pages : 2**

**P810**

# [3621] - 203
# M.A. / M.Sc.
# MATHEMATICS
## MT-603 : Groups and Rings
## (New)

***Time : 3 Hours]*** ***[Max. Marks :80***

***Instructions to the candidates:***

1) ***Attempt any five questions.***
2) ***Figures to the right indicates full marks.***

***Q1)*** a) Define center Z(G) of a group G. Prove that Z(G) is subgroup of G. What is $Z(S_3)$? What is Z(G) if G is abelian group? **[6]**

b) Show that the set H = {5, 15, 25,35} is a group under multiplication modulo 40. What is relation of H with U(8)? Justify. **[5]**

c) Let G = <$a$> with O($a$) = 24. List all the generators of the subgroup of order 8. **[5]**

***Q2)*** a) Prove that every permutation of a finite set can be written as a cycle or product of disjoint cycles. **[6]**

b) If a permutation $\sigma$ is a product of two disjoint cycles of length 4 and 6, What is O($\sigma$)? Find the O($\tau$), where **[5]**

$$\tau = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 7\,6\,1\,2\,3\,4\,5 \end{pmatrix}.$$

c) If $\alpha$ and $\beta$ are a per mutations in $S_n$, show that $\beta\,\alpha\,\beta^{-1}$ and $\alpha$ are both even or both odd. **[5]**

***Q3)*** a) Let $\phi : G \rightarrow \overline{G}$ be a group isomorphism. Show that i) for any $a \in G$, $O(a^{-1}) = O(a)$ and ii) if G is cyclic, $\overline{G}$ is also cyclic. **[6]**

b) Let G be a group and $a \in G$. Show that a map $\phi_a : G \rightarrow G$ defined by $\phi_a(x) = a\,x\,a^{-1}$, $x \in G$ is an isomorphism. What if G is abelian? **[6]**

c) Prove that every group of prime order is cyclic. **[4]**

**P.T.O.**

*Q4)* a) Discuss the classification of groups of order at most 7. **[6]**

b) Let G and H be finite cyclic groups. Then prove that $G \oplus H$ is cyclic if and only if orders of G and H are relatively prime. **[6]**

c) Find all subgroups of order 4 in $Z_4 \oplus Z_4$. **[4]**

*Q5)* a) Let G be a finite abelian group and let $p$ be a prime that divides order of G. Then prove that G has an element of order $p$. **[8]**

b) Prove that if H is a subgroup of a group of index 2 in G, then H is normal in G. **[4]**

c) What is order of $5 + \langle 6 \rangle$ in the factor group $Z_{18} / \langle 6 \rangle$? **[4]**

*Q6)* a) Let G be a finite group whose order is a power of a prime $p$. Then show that Z(G) has more than one element. **[6]**

b) Determine the sylow 3 -subgroup H and sylow 11-subgroup K of G of order 99. Next show that $G = H \times K$. **[6]**

c) Describe the subrings of the ring of integers. **[4]**

*Q7)* a) Prove that intersection of any two subrings is again a subring. What about the union? Justify. **[5]**

b) Let R be a ring and that $a^2 = a$ for all $a \in R$, then show that R is commutative ring. **[5]**

c) Prove that a finite integral domain is a field, hence or otherwise show that Zp is field, where $p$ is prime. **[6]**

*Q8)* a) Let R be a commutative ring with unity. Prove that an ideal M of R is maximal if and only if R/M is field. **[6]**

b) Show that $Z/2Z$ is isomorphic to $Z_2$. Is $Z/2Z$ a field? Justify. **[5]**

c) Show that kernel of a ring homomorphism is an ideal. **[5]**

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Total No. of Questions : 8] [Total No. of Pages : 2

**P810**

# [3621] - 203
# M.A. / M.Sc.
# MATHEMATICS
## MT-603 : Group Theory
## (Old)

*Time : 3 Hours]* *[Max. Marks :80*

***Instructions to the candidates:***

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

***Q1)*** a) Let X be a nonempty set, and let $S_x$ denote the set of all bijective mappings of X into itself. Show that $S_x$ is a group under composition of mappings. Is $S_x$ commutative? Justify your answer. **[8]**

b) Show that for any integer $n$, $x, y \in G$, $(x\,y\,x^{-1})^n = x\,y^n\,x^{-1}$. **[4]**

c) Show that inverse of an element in a group is unique. **[4]**

***Q2)*** a) Prove that no two of the additive groups Z, Q, R are isomorphic to each other. **[4]**

b) For $\alpha, \beta \in S_n$ any permutations, then show that $\beta\,\alpha\,\beta^{-1}$ and $\alpha$ are both even or both odd. **[4]**

c) If $s = \begin{pmatrix} 1, 2 \,........\, n \\ 1'\; 2' \,........\, n' \end{pmatrix}$ and $t$ any permutation in $S_n$, then show that **[4]**

$$tst^{-1} = \begin{pmatrix} t(1)\; t(2) \,........\, t(n) \\ t(1')\; t(2') \,........\, t(n') \end{pmatrix}.$$

d) Show that any two disjoint cycles commute. **[4]**

***Q3)*** a) Define a commutator subgroup G' of a group G. Show that G' is normal subgroup of G, and $G/G'$ is abelian group. **[6]**

b) Prove that two cyclic groups G = $\langle a \rangle$ and G' = $\langle a' \rangle$ are isomorphic if and only if they have the same order. **[6]**

c) If an element $a \in G$, has order $n$, then show that $a^k$ has order $\frac{n}{d}$, where $d = (n, k)$. **[4]**

**P.T.O.**

*Q4)* a) Prove that the relation of conjugacy among the elements of a group G is an equivalence relation. Find the conjugacy classes of the group $S_3$. **[6]**

b) Two permutations $\sigma$ and $\tau$ in $S_n$ are conjugate to each other if and only if they have the same cycle structure. Find the seven conjugacy classes in $S_5$. **[8]**

c) Write down the class equation of a group, explain the terms involved. **[2]**

*Q5)* a) If Z(G) is center of G and $G/Z(G)$ is cyclic, then prove that G is abelian. If G is a group of order $p^2$, $p$-prime, then G is always abelian. **[8]**

b) Prove that every quotient group of an abelian group is abelian. What about converse? Justify. **[4]**

c) Prove in usual notations $C^*/T \approx R^+$, $C^*$ – multiplicative group of non-zero complex numbers, $R^+$ – multiplicative group of positive reals and T – the multiplicative group of complex numbers whose absolute value is 1. **[4]**

*Q6)* a) Define direct product of two subgroups. Suppose A and B are subgroups of a group G, then prove that G = AB (direct) if and only if i) Every element of A commutes with every element of B and ii) Every element $x \in G$ can be written as $x = ab$, $a \in A$, $b \in B$ uniquely. **[10]**

b) Let G be a cyclic group of order 12 generated by $a$. Describe the factors of the series:
$E \subset \langle a^6 \rangle \subset \langle a^3 \rangle \subset \langle a \rangle = G$ and $E \subset \langle a^4 \rangle \subset \langle a^2 \rangle \subset \langle a \rangle = G$, and conclude that the above series are composition series of G. **[6]**

*Q7)* a) Let G be a group of order $n$ and $p$ be a prime such that $p^\beta$ divides $n$, then prove that G has a subgroup of order $p^\beta$. **[10]**

b) Prove that any group of order 99 is direct product of its two non-trivial subgroups. **[6]**

*Q8)* a) Prove that every subgroup and every quotient group of a soluble group is soluble. If H is soluble normal subgroup of a group G and G/H is soluble, prove that G is soluble. **[8]**

b) Show that $S_3$ is soluble. **[4]**

c) If an integer $a$ is prime to a natural number $n$, then prove that $a^{\phi(n)} \equiv 1 \pmod{n}$. **[4]**

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**Total No. of Questions : 4]** **[Total No. of Pages : 2**

**P813**

# [3621] - 206
# M.A. / M.Sc.
# MATHEMATICS
## MT-606 : Object Oriented Programming Using C++
## (Old & New)

***Time : 2 Hours]*** ***[Max. Marks : 50***

***Instructions to the candidates:***

1) *Q1 is compulsory. Attempt any Two questions from Q2, Q3 and Q4.*
2) *Figures to the right indicates full marks.*

***Q1)*** Attempt the following subquestions. **[20]**

a) State true or false:

The && and 11 operators compare two boolean values.

b) State true or false:

A protected member of a base class cannot be accessed from a member function of the derived class.

c) Fill in the blanks:

When the class B is inherited from the class A, class A is called the — class and B is called the —— class.

d) Fill in the blanks:

A pointer to ———— can hold pointers to any data type.

e) Find errors, if any, in the following function prototype: float average $(x, y)$;.

f) What are objects? How are they created?

g) What is a friend function? What are the merits of using friend functions?

h) How do we invoke a constructor function?

i) What is operator overloading?

j) What does inheritance mean in C++?

***Q2)*** a) Write a class to represent a vector. Include member functions to perform the following tasks: **[7]**

i) To create a vector,

ii) To display the vector.

b) Define a class to represent time in hours, minutes and seconds. Include the following member functions: **[8]**

i) To assign initial values,

ii) To display the time,

iii) To add to given times.

***Q3)*** a) Define a class to represent a complex number. Include the following member functions: **[7]**

i) To read a complex number,

ii) To display a complex number,

iii) To add two given complex numbers.

b) Define a class string that could work as a user-defined string type. Include constructor that will enable us to create an initialized string. Include a member function that adds (con catenates) two strings to make a third string. **[8]**

***Q4)*** a) Create a class Float that contains one float data member. Overload all the four arithmetic operators so that they operate on the objects of Float. **[8]**

b) Write a code segment, using nested loops, to display the following output: **[7]**

```
1   2   3   4   5
1   2   3   4
1   2   3
1   2
1
```

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Total No. of Questions : 8] [Total No. of Pages : 3

**P819**

# [3621] - 306
# M.A. / M.Sc.
# MATHEMATICS
## MT-706 : Numerical Analysis
## (Old)

*Time : 3 Hours]* *[Max. Marks :80*

***Instructions to the candidates:***

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*
3) *Use of non-programmable scientific calculators is allowed.*

***Q1)*** a) Investigate the nature of the iteration $p_{k+1} = g(p_k)$, $k \geq 0$ when the function $g(x) = 1 + x - x^2/4$ is used. **[8]**

b) Use three -digit rounding arithmetic to compute the sum :

$$S = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729}.$$ **[8]**

***Q2)*** a) Use Newton's method with starting value $(p_0, q_0) = (2.00, 0.25)$ and compute $(p_1, q_1)$, $(p_2, q_2)$ and $(p_3, q_3)$ for the non-linear system **[8]**

$$x^2 - 2x - y + 1/2 = 0,$$
$$x^2 + 4y^2 - 4 = 0.$$

b) Suppose that Newton-Raphson iteration produces a sequence $\{p_n\}_{n=0}^{\infty}$ that converges to the root $p$ of the function $f(x)$. If $p$ is a simple root, prove that $|e_{n+1}| \approx \left[\frac{|f''(p)|}{2|f'(p)|}\right] |e_n|^2$, for large $n$. **[8]**

***Q3)*** a) Use Gaussian elimination to construct the triangular factorization of the matrix $\begin{bmatrix} 4 & 3 & -1 \\ -2 & -4 & 5 \\ 1 & 2 & 6 \end{bmatrix}$. **[8]**

b) Suppose A is a diagonally dominant matrix. Prove that the sequence of Jacobi iterations $\{x^{(k)}\}$ converges to the unique solution of the system $Ax = b$ for any choice of starting vector $x^{(0)}$. **[8]**

*Q4)* a) Solve the system $Ax = b$ where $b = [-4, 10, 5]^T$ and A has the following decomposition: $A = \begin{bmatrix} 2 & 4 & -6 \\ 1 & 5 & 3 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -6 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ **[8]**

b) Assume that $f \in c^{N+1}[a, b]$ and that $x_0, x_1, \ldots\ldots x_N \in [a, b]$. If $x \in [a, b]$, then prove that the error term in the lagrange polynomial approximation $f(x) = P_N(x) + E_N(x)$ is given by $E_N(x) = \prod_{j=0}^{N} (x - x_j) f^{(N+1)}(c)/(N+1)!$ for some $c \in [a, b]$. **[8]**

*Q5)* a) Let $f(x) = x^3 - 4x$. Construct divided difference tabel based on the nodes $x_0 = 1, x_1 = 2, \ldots\ldots, x_5 = 6$, and find the Newton polynomial $P_3(x)$ based on $x_0, x_1, x_2, x_3$. **[8]**

b) Assume that $f \in c^3[a, b]$ and that $x - h, x, x + h \in [a, b]$. Derive the central difference formula: $f'(x) \approx [f(x + h) - f(x - h)]/2h$. **[8]**

*Q6)* a) Let $f(x) = \cos x$. Use the following formula

$$f'(x) \approx [-f(x + 2h) + 8f(x + h) - 8f(x - h) + f(x - 2h)] / 12h$$

with $h = 0.0001$ and calculate $f'(0.8)$. **[8]**

b) Use Euler's method to solve the initial value problem $2y' = t - y$ on $[0, 3]$, $y(0) = 1$ with $h = 1$. **[8]**

*Q7)* a) Solve the following initial value problem by using Runge-Kutta method of order 4: **[8]**

$y' = t^2 - y$ over $[0, 0.2]$, $y(0) = 1$ with $h = 0.1$.

b) Suppose that $[a, b]$ is subdivided into M subintervals $[x_k, x_{k+1}]$ of width $h$. Derive the composite Trapezoidal rule in the form. **[8]**

$$\int_a^b f(x)dx = \frac{h}{2}[f(a) + f(b)] + h\sum_{k=1}^{M-1} f(x_k).$$

*Q8)* a) Start with $x_0 = [1, 1, 1]^T$ and use power method to find the dominant eigen pair for the matrix : $\begin{bmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{bmatrix}$. Perform three iterations. **[8]**

b) Define a Householder matrix and prove that it is always symmetric. **[4]**

c) Define an orthogonal matrix and show that the following matrix is orthogonal: $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ for any fixed $\theta$. **[4]**

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**Total No. of Questions : 8]** **[Total No. of Pages :3**

**P803**

# [3621] - 101

## M.A. / M.Sc. (New)

## MATHEMATICS

## MT - 501 : Real Analysis

***Time : 3 Hours]*** ***[Max. Marks :80***

***Instructions to the candidates:***

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

***Q1)*** a) If (V, $\langle \cdot , \cdot \rangle$) is an inner product space and $\|v\|$ is defined by

$$\|v\| = \sqrt{\langle v , v \rangle}$$

then show that $\|\cdot\|$ is a norm on V. **[5]**

b) If C[$a$, $b$] denotes the set of all complex valued continuous functions on [$a$, $b$], then show that $\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\, dx$ is an inner product on C [$a$, $b$]. **[6]**

c) Prove that in any complex inner product space $\langle v, \lambda w\rangle = \bar{\lambda}\langle v, w\rangle$ for every v, w $\in$ V and each $\lambda \in$ C. **[5]**

***Q2)*** a) State and prove the Heine-Borel Theorem. **[8]**

b) Is it true that int (A $\cup$ B) = int (A) $\cup$ int (B)? Prove or give a counter example. **[4]**

c) Consider R with Euclidean metric

i) Find the set of limit points of

$$\left\{\frac{1}{n} \mid n = 1, 2, 3, .......\right\}.$$

ii) Find the set of limit points of Q. **[4]**

***Q3)*** a) State and prove Ascoli - Arzela's Theorem. **[8]**

b) Show that C ([$a$, $b$] ;R) with the supremum norm is complete. **[6]**

c) Show that R with discrete metric is not separable. **[2]**

***P.T.O.***

*Q4)* a) For an interval I = $[a_1, b_1]$ X .... X $[a_n, b_n]$ in $R^n$ define

$$m\ (I) = \prod_{k=1}^{n} (b_k - a_k).$$

Let ε is the collection of all finite unions of disjoint intervals in $R^n$. Show that $m$ is a measure on ε. **[6]**

b) Assume that $f \geq 0$ is measurable, and that $A_1, A_2, \ldots\ldots, \ldots \in M$ are pairwise disjoint. Then show that

$$\int_{\bigcup_{k=1}^{\infty} A_k} f dm = \sum_{k=1}^{\infty} \left( \int_{A_k} f\, dm \right)$$

(with usual notations). **[7]**

c) If $f$ is measurable then show that $|f|$ is measurable. **[3]**

*Q5)* a) State and prove Lebesgue's Monotone convergence Theorem. **[8]**

b) State Lebesgue's Dominated Convergence Theorem. **[2]**

c) For $1 \leq p < \infty$. Show that $L^p$ (μ) is a normed linear space, with the norm given by

$$\|f\|_p = \left( \int_X |f|^p\, d\mu \right)^{1/p}.$$ **[6]**

*Q6)* a) Prove that in any inner product space (V, ⟨ , ⟩), $f$ & $g$ are orthogonal implies that

$$\| f \|^2 + \| g \|^2 = \| f + g \|^2.$$ **[5]**

b) Show that the trigonometric system

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos\ (nx)}{\sqrt{\pi}}, \frac{\sin\ (mx)}{\sqrt{\pi}},$$

$n, m$ = 1, 2, 3,------ is an orthonormal sequence in $L^2$ ([–π , π], $m$). **[5]**

c) Suppose that $f = \sum_{k=1}^{\infty} c_k f_k$ for an orthonormal sequence $\{f_k\}_{k=1}^{\infty}$ in an inner product space V. Then show that $c_k = \langle f, f_k \rangle$ for each $k$. **[6]**

*Q7)* a) Show that the classical Fourier series of $f(x) = x$ is

$$2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$ **[5]**

b) If $\{U_n\}_{n=1}^{\infty}$ is a sequence of open dense subsets of a complete metric space M then $\bigcap_{n=1}^{\infty} U_n$ is dense in M. **[6]**

c) Show that X is nowhere dense in a metric space (M, $d$) if and only if its closure has empty interior. **[5]**

*Q8)* a) State and prove the Weierstrass Approximation Theorem. **[12]**

b) Give statements of **[4]**

i) Fatou's Lemma.

ii) Stone - Weierstrass theorem.

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**Total No. of Questions : 8]** **[Total No. of Pages :2**

**P803**

# [3621] - 101
# M.A. / M.Sc. (Old)
# MATHEMATICS
## MT - 501 : Real Analysis - I

***Time : 3 Hours]*** ***[Max. Marks :80***

***Instructions to the candidates:***

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

***Q1)*** a) Define a countable and uncountable sets and show that R/Q, the set of irrational numbers, is uncountable. **[6]**

b) If A is an infinite subset of N then show that A is equivalent to N, where N is the set of all natural numbers. **[5]**

c) State and prove Cantor's theorem. **[5]**

***Q2)*** a) Define normed linear space and show that for any $x, y \in l_2$,

$$\sum_{i=1}^{\infty} |x_i\, y_i| \leq \|x\|_2\, \|y\|_2 .$$ **[8]**

b) Prove that Any subset of a metric space is again a metric space. **[8]**

***Q3)*** a) If U is an open subset of R, then prove that U may be written as a countable union of disjoint open intervals. That is $U = \bigcup_{n=1}^{\infty} I_n$.

where $I_n = (a_n, b_n)$ and $I_n \cap I_m = \phi$ for $n \neq m$. **[8]**

b) Let $A \subset (M, d)$, then prove that $x \in \overline{A}$ iff $B_\varepsilon(x) \cap A \neq \phi$ for every $\varepsilon > 0$. **[6]**

c) Define the following terms with an example. **[2]**

i) Separable space.

ii) Isolated point.

***Q4)*** a) Prove that M is disconnected iff there exist a continuous map from M <u>onto</u> $\{0, 1\}$. **[6]**

b) If E and F are connected subsets of M with $E \cap F \neq \phi$ then prove that $E \cup F$ is connected. **[5]**

***P.T.O.***

c) Let $f : (M, d) \to (N, \rho)$ be continuous, and let E be a subset of M. If E is connected then prove that $f(E)$ is connected. **[5]**

***Q5)*** a) Let (M, $d$) be a complete metric space and let A be a subset of M. Then prove that (A, $d$) is complete iff A is closed in M. **[6]**

b) Give an example which shows that boundedness doesn't imply totally boundedness. **[5]**

c) State and prove Bolzano-Weierstrass Theorem. **[5]**

***Q6)*** a) If M is compact metric space, then prove that every continuous map $f : M \to N$ is uniformly continuous. **[6]**

b) Let $f : (M, d) \to (N, \rho)$ be continuous and K is compact in M then prove that $f(K)$ is compact in M. **[6]**

c) Define the following terms with an example **[4]**

i) Compact metric space.

ii) Uniform continuity.

***Q7)*** a) State and prove Baire category theorem. **[8]**

b) Let (X, $d$) and (Y, $\rho$) be metric space, and let $f$ and $\{f_n\}$ be functions mapping X into Y. If $(f_n)$ converges uniformly to '$f$' on X and if each $f_n$ is continuous at $x \in X$ then prove that $f$ is also continuous at $x$. **[8]**

***Q8)*** a) Prove that C [0, 1] is separable. **[7]**

b) State and prove Arzela - Ascoli Theorem. **[7]**

c) State the Weierstrass approximation theorem. **[2]**

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**Total No. of Questions : 8]** **[Total No. of Pages :3**

**P805**

# [3621] - 103

# M.A. / M.Sc.

# MATHEMATICS

## MT - 503 : Linear Algebra (New)

***Time : 3 Hours]*** ***[Max. Marks :80***

***Instructions to the candidates:***

1) ***Answer any five questions.***
2) ***Figures to the right indicate full marks.***
3) ***F. D. V. S. - finite dimensional vector space.***
4) ***L (V, V') - set of all vector space linear transformations from V to V'.***

***Q1)*** a) Let V be F. D. V. S. over K. Then prove that the following statements are equivalent for a subset B of V. **[6]**

i) B is a basis.

ii) B is a minimal generating set, that is no subset of B can generate V.

iii) B is a maximal linearly independent set.

b) Let $W = \{ [x_1 \;\; x_2 \;\; x_3 \;\; x_4 ]^t \in R^4 \mid 2x_1 + 3x_2 = 4x_3 + x_4\}$. Show that W is a subspace of $R^4$. Find a basis of W and extend it to form a basis of $R^4$. **[6]**

c) Prove that if char K $\neq$ 2, then a bilinear form on V over K is skew-symmetric if and only if it is alternating. **[4]**

***Q2)*** a) Let V and V' be vector spaces over K and let $T \in L(V, V')$. If W and W' are subspaces of V and $V'$ respectively and $T(W) \subseteq W'$, prove that T induces a linear transformation $\overline{T} : V/W \to V'/W'$ defined by $\overline{T}(v + w) = T(v) + w'$. Also, if T is surjective, prove that $\frac{V}{\ker T} \simeq V'$. **[7]**

b) Let V be a finite dimensional vector space over K and let W be a subspace of V. Then prove that dim V = dim W + dim V/W. **[5]**

c) Let $\text{Sym} = \{A \in K^{n\times n} \mid A^t = A\}$ and

$\text{SSym} = \{A \in K^{n\times n} \mid A^t = -A\}$,

be the subspaces of $K^{n\times n}$ where K is a field with char K $\neq$ 2. Prove that $K^{n\times n} = \text{Sym} \oplus \text{SSym}$. **[4]**

***P.T.O.***

*Q3)* a) Let V be a F. D. V. S. over K. Prove that the mapping $e : V \rightarrow V^{**}$ defined by $v \rightarrow e_v$ is an isomorphism, where $e_v$ is the evaluation mapping and $V^{**}$ denotes the dual space of V. **[6]**

b) Let D be the differential operator on $R_3[x]$ write the matrix representation of T with respect to the ordered basis $\{1-x, 1+x^2, x-x^3, -x^2+x^3\}$. **[6]**

c) Show that if $\lambda$ is an eigen value of T and $P(x) \in K[x]$, then $P(\lambda)$ is an eigenvalue of P(T). **[4]**

*Q4)* a) Let T be the linear operator on F. D. V. S. V over K. If T is diagonalizable, prove that the characteristic polynomial of T splits over K and the algebraic multiplicity of each eigenvalue equals its geometric multiplicity. **[7]**

b) For the matrix A, where $A = \begin{bmatrix} 3 & -2 & 2 \\ 10 & -9 & 10 \\ 6 & -6 & 7 \end{bmatrix}$ verify that 1 is an eigenvalue and find its geometric and algebraic multiplicities. **[5]**

c) Prove that a Jordan chain consists of linearly independent vectors. **[4]**

*Q5)* a) Prove that two triangulable $n \times n$ matrices are similar if and only if they have the same Jordan canonical form. **[6]**

b) Find the rational canonical form for A, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix} \in Q^{3\times 3}.$$ **[5]**

c) Show that for $x, y$ in an inner product space V over F, $\|x+y\| = \|x\| + \|y\|$ if and only if $x = 0$ or $y = \lambda x$, for some $\lambda \in F$. **[5]**

*Q6)* a) Let V be a finite dimensional inner product space and let $f$ be a linear functional on V. Prove that there exists a unique vector $x$ in V such that $f(v) = (v, x)$ for all $v \in V$. **[7]**

b) Let $A \in F^{n\times n}$. Then prove that A is a unitary matrix iff the columns of A form an orthonormal basis of the standard inner product space $F^n$. **[5]**

c) Let T be a self adjoint operator on an inner product space V. If V is finite dimensional, then prove that all roots of characteristic polynomial of T are real. **[4]**

*Q7)* a) Let T be a self adjoint operator on a finite dimensional inner product space V. Prove that T is positive definite if and only if all eigenvalues of T are positive. **[6]**

b) Prove that a linear operator T on a finite dimensional inner product space V can be represented in the form T = UP where U is unitary and $P = (T^*T)^{1/2}$, a positive semidefinite operator which is uniquely determined by T. **[6]**

c) If W is subspace of a finite dimensional inner product space, then prove that $V = W \oplus W^{\perp}$. Where $W^{\perp}$ is the orthogonal complement of W. **[4]**

*Q8)* a) Let T be a triangulable linear operator on a finite dimensional inner product space V over F. Prove that T is normal if and only if V has an orthonormal basis consisting of eigenvectors of T. **[6]**

b) Prove that a symmetric bilinear form on a finite dimensional vector space V over a field K of characteristic not equal to 2 is diagonalizable. **[6]**

c) Let V be a vector space over K, char $K \neq 2$. Prove that a bilinear form on V can be uniquely written as a sum of symmetric and skew-symmetric bilinear forms. **[4]**

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**Total No. of Questions : 8]** **[Total No. of Pages :3**

**P805**

# [3621] - 103

# M.A. / M.Sc.

# MATHEMATICS

## MT - 503 : Linear Algebra (Old)

***Time : 3 Hours]*** ***[Max. Marks :80***

***Instructions to the candidates:***

1) ***Answer any five questions.***
2) ***Figures to the right indicate full marks.***
3) ***F. D. V. S : Finite dimensional vector space.***
4) ***A(V) : Algebra of linear transformations on V.***

***Q1)*** a) If V and W are vector spaces of dimensions $m$ and $n$ respectively over F, then prove that Hom (V, W), the set of all vector space homomorphisms of V into W, is of dimension $mn$ over F. **[7]**

b) If F is the field of real numbers, find Ann(W), the annihilator of W, where W is spanned by (0, 0, 1, –1), (2, 1, 1, 0) and (2, 1, 1, –1). **[4]**

c) If T is a homomorphism of modules M into N, prove that $K(T) = \{x \in M \mid xT = 0\}$ is a submodule of M and $I(T) = \{xT \mid x \in M\}$ is a submodule of N. **[5]**

***Q2)*** a) If V is F. D. V. S. over F, prove that $T \in A(V)$ is invertible iff the constant term of the minimal polynomial for T is not 0. **[6]**

b) If V is F. D. V. S. over F, prove that $T \in A(V)$ is regular iff T maps V onto V. **[6]**

c) If $T \in A(V)$ is nilpotent, and if $vT = \alpha v$ for some $v \neq 0$ in V, with $\alpha \in F$, prove that $\alpha = 0$. **[4]**

***Q3)*** a) If $T, S \in A(V)$ and if S is regular, prove that T and $STS^{-1}$ have the same minimal polynomial. **[6]**

b) Let V be the vector space of polynomials of degree 3 or less over F. In V, define T by, $(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)T = \alpha_0 + \alpha_1 (x + 1) + \alpha_2 (x + 1)^2 + \alpha_3 (x + 1)^3$. Compute the matrix of T in the basis $1, 1+x, 1+x^2, 1+x^3$. **[6]**

c) If $\lambda \in F$ is a characteristic root of $T \in A(V)$, prove that for any polynomial $q(x) \in F[x]$, $q(\lambda)$ is a characteristic root of $q(T)$. **[4]**

***P.T.O.***

*Q4)* a) If $T \in A(V)$ has all its characteristic roots in F, prove that there is a basis of V in which the matrix of T is triangular. **[7]**

b) If V is n-dimensional over F and if $T \in A(V)$ has all its characteristic roots in F, prove that T satisfies a polynomial of degree $n$ over F. **[5]**

c) Let $T \in A(V)$ and $\lambda \in F$ be a characteristic root of T in F. Let $U_\lambda = \{v \in V \mid vT = \lambda v\}$. If $S \in A(V)$ commutes with T, prove that $U_\lambda$ is invariant under S. **[4]**

*Q5)* a) If $T \in A(V)$ has a minimal polynomial $p(x) = q(x)^e$, where $q(x)$ is a monic, irreducible polynomial in $F[x]$, prove that a basis of V over F can be found in which the matrix of T is of the form

$$\begin{bmatrix} C(q(x)^{e_1}) & & & \\ & C(q(x)^{e_2}) & & \\ & & \ddots & \\ & & & C(q(x)^{e_r}) \end{bmatrix}$$

where $e = e_1 \geq e_2 \geq .... \geq e_r$ and $C(f(x))$ denotes companion matrix of $f(x)$. **[7]**

b) Prove that two nilpotent linear transformations, having same invariants, are similar. **[5]**

c) Find the invariants and Jordan form of matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$ **[4]**

*Q6)* a) If field F is of characteristic 0, and if S and T in $A_F(V)$ are such that ST–TS commutes with S, then prove that ST– TS is nilpotent. **[6]**

b) Prove that the determinant of a triangular matrix is the product of its entries on the main diagonal. **[6]**

c) If T is nilpotent transformation, and tr T denotes trace of T, prove that $\text{tr } T^i = 0$ for all $i \geq 1$. Is the converse true? Justify. **[4]**

*Q7)* a) If $T \in A(V)$ is such that $(vT, v) = 0$ for all $v \in V$, prove that $T = 0$. **[6]**

b) If $T \in A(V)$ is Hermitian, prove that all its characteristic roots are real. **[6]**

c) If T is unitary and $\lambda$ is a characteristic root of T, prove that $|\lambda| = 1$. **[4]**

***Q8)*** a) If $\lambda$ is a characteristic root of the normal transformation N and if $vN = \lambda v$, then prove that $vN^* = \bar{\lambda} v$. **[6]**

b) If T is Hermitian and $vT^k = 0$ for $k \geq 1$, prove that $vT = 0$. **[5]**

c) Determine the rank and signature of the real quadratic form $x_1^2 + 2x_1 x_2 + x_2^2$. **[5]**

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**Total No. of Questions : 8]** **[Total No. of Pages : 2**

**P807**

# [3621]- 105
# M.A./M.Sc.
# MATHEMATICS
## MT - 505 : Ordinary Differential Equations
## (Old & New)

***Time : 3 Hours]*** ***[Max. Marks : 80***

***Instructions to the candidates:***

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

***Q1)*** a) If $\phi_1(x)$ and $\phi_2(x)$ are two linearly independent solutions of the equation $y'' + P(x)y' + Q(x)y = 0$ then prove that the zeros of these functions are distinct and occur alternatly. **[8]**

b) Show that the general solution of the constant coefficient equation $y'' + Py' + Qy = 0$ tends to zero as $x \to \infty$ if and only if the real parts of the roots of the associated polynomial are negative. **[8]**

***Q2)*** a) Find the general solution of $y'' + 4y = \cos x$. **[6]**

b) Are the functions $\phi_1(x) = x$ and $\phi_2(x) = |x|$ defined on $-\infty < x < \infty$ linearly independent? Why? **[4]**

c) Verify that $\phi_1(x) = x^2$ is one solution of the equation $x^2 y'' + xy' - 4y = 0$, and then find $\phi_2(x)$ and the general solution. **[6]**

***Q3)*** a) Let $\phi_1(x), \phi_2(x)$ be two solutions of $y'' + P(x)y' + Q(x)y = 0$. Then show that Wronskian W of $\phi_1$ and $\phi_2$ is either nowhere vanishing or vanishing identically. **[6]**

b) Find the eigen values $\lambda_n$ and eigen functions $\phi_n(x)$ for the equation $y'' + \lambda y = 0, y(a) = y(b) = 0, \ a < b$. **[6]**

c) Show that $y(x) = c_1 e^x + c_2 e^{2x}$ is the general solution of the equation $y'' - 3y' + 2y = 0$. Find the particular solution for which $y(0) = -1$ and $y'(0) = 1$. **[4]**

***Q4)*** a) Express the function arc sin $x$ in the form of a power series $\sum_j a_j x^j$ by solving the differential equation $y' = (1 - x^2)^{\frac{-1}{2}}$ in two ways. Hence obtain $\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3.2^3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5.2^5} + .......$ **[8]**

b) Verify that the origin is a regular singular point and calculate two independent Frobenius series solutions for the differential equation. $2x^2 y'' + x(2x + 1)y' - y = 0$. **[8]**

*Q5)* a) Show that $-1$ and $1$ are regular points for the Legendre equation $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$. Also find the indicial polynomial, and its roots corresponding to $x = 1$. **[8]**

b) Find the general solution of $(1 - e^x)y'' + \frac{1}{2}y' + e^x y = 0$. Near the singular point $x = 0$. **[8]**

*Q6)* a) Determine the nature of the point at infinity for the equations : **[8]**

i) $x^2 y'' + xy' + (x^2 - \alpha^2)y = 0$

ii) $x^2 y'' + 2xy' - n(n+1)y = 0$

b) Consider the generating relation **[8]**

$(1 - 2xt + t^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} p_n(x)\, t^n$ for the Legendre polynomials and obtain recurrence formula $(1-x^2)\, p_n^1(x) = n\,[p_{n-1}(x) - x\, p_n(x)]$.

*Q7)* a) Find the general solution of the system $\frac{dx}{dt} = 3x - 4y,\ \frac{dy}{dt} = x - y$. **[6]**

b) If $a_1 b_2 - a_2 b_1 \neq 0$, show that the system

$\frac{dx}{dt} = a_1 x + b_1 y,\ \frac{dy}{dt} = a_2 x + b_2 y$ has infinitely many critical points, none of which are isolated. **[6]**

c) Show that the function $f(x, y) = y^{\frac{1}{2}}$ satisfy a Lipschitz condition on any rectangle $R : |x| \leq a,\ b \leq y \leq c,\ (a, b, c > 0)$. **[4]**

*Q8)* a) Prove that a function $\phi$ is a solution of the initial value problem $y' = f(x, y),\ y(x_0) = y_0$ on an interval I if and only if it is a solution of the integral equation $y = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$ on I.

**[4]**

b) Solve the following initial value problem by Picard's method and compare the result with exact solution. **[8]**

$$\frac{dy}{dx} = 3y + 1,\quad y(0) = 2$$

c) Replace the following differential equation by an equivalent system of first order equation.

$y''' + ay'' + by' + cy = 0$. **[4]**

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**Total No. of Questions : 8]** **[Total No. of Pages : 3**

# P809

## [3621] - 202

## M.A. / M.Sc.

## MATHEMATICS

## MT - 602 : Differential Geometry

## (Old & New)

***Time : 3 Hours]*** ***[Max. Marks : 80***

***Instructions to the candidates:-***

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

***Q1)*** a) Define : **[4]**

i) Level sets.

ii) Graph of function.

b) Sketch the level sets $f^{-1}(c)$, for $n = 0,1$ of the following function at the height indicated. **[6]**

$f(x_1, x_2, \ldots, x_{n+1}) = x_1^2 + x_2^2 + \ldots\ldots x_{n+1}^2$, $c = 1, 2, 4$.

c) Find the integral curve through $p = (1, 1)$ of the vector field $X(x_1, x_2) = (x_2, -x_1)$. **[6]**

***Q2)*** a) Let U be an open set in $R^{n+1}$ and let $f : U \to R$ be a smooth function. Let $p \in U$ be a regular point of $f$, and let $c = f(p)$. Prove that the set of all vectors tangent to $f^{-1}(c)$ at $p$ is equal to $[\nabla f(p)]^{\perp}$. **[8]**

b) Show by example that the set of vectors tangent at a point $p$ of a level set might be all of $R_p^{n+1}$. **[4]**

c) Show that the unit n-sphere $x_1^2 + x_2^2 + \ldots\ldots x_{n+1}^2 = 1$ is an n-surface. **[4]**

***Q3)*** a) Let $a, b, c, \in R$ be such that $ac - b^2 > 0$. Show that the maximum and minimum values of the function $g(x_1, x_2) = x_1^2 + x_2^2$ on the ellipse $ax_1^2 + 2bx_1x_2 + cx_2^2 = 1$ are of the form $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$, where $\lambda_1$ and $\lambda_2$ are the eigenvalues of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. **[6]**

b) Let $S \subset R^{n+1}$ be a connected n-surface in $R^{n+1}$. Prove that there exist on S exactly two smooth unit normal vector fields $N_1$ and $N_2$ and $N_2(p) = -N_1(p)$ for all $p \in S$. **[6]**

c) Find two different orientations on the n-sphere $x_1^2 + x_2^2 + ........+ x_{n+1}^2 = r^2$ of radius $r > 0$. **[4]**

***Q4)*** a) Define Gauss map and spherical image and find the spherical image of the sphere $x_1^2 + x_2^2 + x_3^2 = r^2$ (r > 0). **[8]**

b) Show that if $\alpha : I \to R^{n+1}$ is a parameterized curve with constant speed then $\ddot{\alpha}(t) \perp \dot{\alpha}(t)$ for all $t \in I$. **[4]**

c) Let S denote the cylinder $x_1^2 + x_2^2 = r^2$ of radius $r > 0$ in $R^3$. Show that $\alpha$ is a geodesic of S, where $\alpha$ is given by

$\alpha(t) = (r\cos(at + b), r\sin(at + b), ct + d)$ for some $a, b, c, d \in R$. **[4]**

***Q5)*** a) Define Levi-Civita parallelism and if X is parallel along $\alpha$, then prove that X has constant length. **[4]**

b) Let S be an n-surface in $R^{n+1}$, let $p, q \in S$, and let $\alpha$ be a piecewise smooth parameterized curve from $p$ to $q$. Prove that parallel transport $p_\alpha : S_p \to S_q$ along $\alpha$ is a vector space isomorphism which preservs dot product. **[6]**

c) Let S be an n-surface in $R^{n+1}$ oriented by the unit normal vector field N. Let $p \in S$ and $V \in S_p$. Prove that for every parameterized curve $\alpha : I \to S$, with $\dot{\alpha}(t_0) = V$ for some $t_0 \in I$,

$$\ddot{\alpha}(t_0).N(p) = L_p(V).V,$$

where $L_p$ is a Weingartan map. **[6]**

***Q6)*** a) Prove that local parameterizations of plane curves are unique up to reparameterization. **[8]**

b) Let $\alpha(t) = (x(t), y(t))$ $(t \in I)$ be a local parameterization of the oriented plane curve C. Show that $K = \dfrac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$. **[4]**

c) Define a differential 1-form and give one example. **[4]**

*Q7)* a) Define a parameterized n-surface and give one example. **[4]**

b) Prove that on each compact oriented n-surface S in $R^{n+1}$ there exists a point $p$ such that the second fundamental form at $p$ is definite. **[8]**

c) Find the Gaussian curvature $K : S \to R$ where S is the surface given by

$$\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}=1\,(a,\ b \text{ and } c \text{ all} \neq 0).$$ **[4]**

*Q8)* a) Let S be an n-surface in $R^{n+1}$ and let $p \in$ S. Prove that there exists an open set V about $p$ in $R^{n+1}$ and a parameterized n-surface $\phi : U \to R^{n+1}$ such that $\phi$ is a one to one map from U onto $V \cap S$. **[8]**

b) Define a diffeomorphism and state and prove inverse function theorem for n-surfaces. **[8]**

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**Total No. of Questions : 8]** **[Total No. of Pages : 3**

**P811**

# [3621]- 204
## M.A./M.Sc.
## MATHEMATICS
## MT - 604 : Complex Analysis
## (New & Old)

***Time : 3 Hours]*** ***[Max. Marks : 80***

***Instructions to the candidates:***

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

***Q1)*** a) Let R be the radius of convergence of the power series $\sum a_n z^n$. Show that $R = \lim \left|\frac{a_n}{a_n + 1}\right|$ if this limit exists. **[6]**

b) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{1}{2^n} z^{2n}$. **[5]**

c) Let G be either the whole plane C or some open disc. If $u : G \to R$ is a harmonic function then show that $u$ has a harmonic conjugate. **[5]**

***Q2)*** a) Define a branch of logarithm. Let $f$ be a branch of logarithm on an open connected subset G of C. Show that the totality of branches of log $z$ are the functions $f(z) + 2\pi ik$, $k \in Z$. **[6]**

b) Let D be the unit disc in C . Let $f : D \to R$ be an analytic function. Show that $f$ is constant. **[5]**

c) Let $f(z) = |z|^2 = x^2 + y^2$ for $z = x + iy \in$ C. Show that $f$ is continuous. Also show that $f$ does not have primitive. **[5]**

***Q3)*** a) Let $z_1, z_2, z_3, z_4$ be four distinct points in $C_\infty$. Show that the cross ratio $(z_1, z_2, z_3, z_4)$ is a real number if and only if all four points lie on a circle. **[6]**

b) Find the fixed points of a dilation, a translation and the inversion on $C_\infty$. **[5]**

c) Under stereographic projection, which subsets of the unit sphere $S^2 \subseteq R^3$ corresponds to real and imaginary axes in C ? **[5]**

*Q4)* a) Let $f$ be analytic in B($a$, R). Show that $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ for $|z - a| < R$, where $a_n = \frac{1}{n!} f^{(n)}(a)$ and this series has radius of convergence $\geq$ R. **[6]**

b) Evaluate the integral $\int_{\gamma} \frac{dz}{z^2+1}$, where $\gamma(t) = 2\,e^{it}$, $0 \leq t \leq 2\pi$. **[5]**

c) Identify all entire functions $f$ such that $|f'(z)| < |f(z)|\ \forall\ z \in C$ ? **[5]**

*Q5)* a) Let $\gamma$ be a closed rectifiable curve in C. Show that $n(\gamma; a)$ is constant for 'a' belonging to a component of $G = C - \{\gamma\}$. **[6]**

b) Let $f: C \to C$ be an analytic function such that $|f(z)| \leq |z|^4\ \forall\ z \in C$. Show that $f$ is a polynomial of degree $\leq 4$. **[5]**

c) Let G be a region and suppose $f_n : G \to C$ is analytic for each $n \geq 1$. Suppose that $\{f_n\}$ converges uniformly to a function $f: G \to C$. Show that $f$ is analytic. **[5]**

*Q6)* a) State and prove Morera's theorem. **[6]**

b) Let G be simply connected and let $f: G \to C$ be an analytic function such that $f(z) \neq 0$ for any $z \in G$. Show that there is an analytic function $g: G \to C$ such that $f(z) = \exp g(z)$. If $z_0 \in G$ and $e^{w_0} = f(z_0)$, then show that we may choose of such taht $g(z_0) = w_0$. **[5]**

c) Let $\gamma(\theta) = \theta\, e^{i\theta}$ for $0 \leq \theta \leq 2\pi$ and $\gamma(\theta) = 4\pi - \theta$ for $2\pi \leq \theta \leq 4\pi$.

Evaluate $\int_{\gamma} \frac{dz}{z^2+\pi^2}$. **[5]**

*Q7)* a) Show that if $f$ has an isolated singularity at '$a$' then the point $z = a$ is a removable singularity if and only if $\lim_{z \to a} (z-a) f(a) = 0$. **[6]**

b) Evaluate the integral $\int_{-\pi}^{\pi} \frac{d\theta}{3+\cos\theta}$. **[5]**

c) Determine the nature of singularities of $\exp\left(\frac{1}{z}\right)$. Find the image of the set $\left\{z : 0 < |z| < \frac{1}{10}\right\}$ under $f$. **[5]**

*Q8)* a) State and prove Schwarz's lemma. **[6]**

b) Give a proof of the fundamental theorem of algebra using Rouche's theorem. **[5]**

c) Let G be a bounded region and suppose $f$ is continuous on $\overline{G}$ and analytic on G. Show that if there is a constant $c \geq 0$ such that $|f(z)| = c$ for all z on the boundary of G then either $f$ is a constant or $f$ has a zero in G. **[5]**

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Total No. of Questions : 8] [Total No. of Pages : 2

**P812**

# [3621] - 205
# M.A. / M.Sc.
# MATHEMATICS
# MT - 605 : Partial Differential Equations
# (Old)

*Time : 3 Hours]* *[Max. Marks : 80*

***Instructions to the candidates:-***

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

***Q1)*** a) If $a$ to $g$ are real functions of $x$ and $y$, find the solution of the equation
$a\,u_{xx} + 2b\,u_{xy} + c\,u_{yy} + d\,u_x + e\,u_y + f\,u = g$
when Cauchy's conditions are imposed on some curve C defined by $y = y(x)$. **[8]**

b) Reduce the equation **[8]**
$$u_{xx} + 5u_{xy} + 6u_{yy} = 0$$
to canonical form.

***Q2)*** a) If $c$ is a constant, obtain the function $u$ which satisfies the one dimensional wave equation **[8]**
$$c^2u_{xx} - u_{tt} = 0.$$

b) By factorizing the differential operator, obtain the general solution of the equation $6u_{xx} + 11u_{xy} + 3u_{yy} = 0$. **[8]**

***Q3)*** a) Suppose that D is a simple closed region bounded by a smooth curve C. Prove that there is at most one solution of Laplace's equation in D and taking prescribed values on C. **[8]**

b) Solve Laplace's equation in $0 < x < a$, $0 < y < b$ when $u(0, y) = 0 = u_x(a, y)$, $u(x, 1) = 1$, $u(x, b) = 0$. **[8]**

***Q4)*** a) Prove that the value of a harmonic function at a point P is equal to the mean of its values on any circle with centre at P. **[8]**

b) Solve, in the square $0 < x < 1$, $0 < y < 1$, $u_{xx} + u_{yy} = g$ with $u = 0$ on the boundary of the square. **[8]**

***P.T.O.***

***Q5)*** a) If F is some function, determine the function $u$ satisfying the equation $u_{xy} = F(x, y, u, u_x, u_y)$ **[8]**

such that on a curve which is nowhere characteristic $u_x = p(x)$, $u_y = q(y)$, $u \equiv u_1(x) \equiv u_2(y)$.

b) Find the solution of $u_{xy} + \lambda u = 0$ **[8]**

with $u(x, 0) = u(0, y) = 1$, $x \geq 0$, $y \geq 0$.

***Q6)*** a) Find the Riemann function appropriate to equation $u_{xy} + N(u_x + u_y)(x + y)^{-1} = 0$, with $N = 2$. **[8]**

b) Solve the following Cauchy problem by finding the appropriate Riemann function $(x - y)u_{xy} - (u_x - u_y) = 0$,

$u = 0$, $u_x = 2x^2$ on $x + y = 0$. **[8]**

***Q7)*** a) Prove that the values within the rectangle R defined by $0 \leq x \leq a$, $0 \leq t \leq T$ of a continuous function $u(x, t)$ satisfying the homogeneous equation

$P_2 u = u_{xx} - u_t = 0$

do not exceed the maximum value $u$ on the lines $t = 0$, $x = 0$, $x = a$. **[8]**

b) A solution $u$ of the heat equation in $0 < x < a$, $t > 0$ is such that on $x = 0$, $x = a$, and $t = 0$, $A < u < B$, where A and B are constants. Show that $A < u < B$ for $0 < x < a$, $t > 0$. **[8]**

***Q8)*** a) Reduce the system **[8]**

$$u_y^{(1)} + 4u_x^{(2)} = 0$$

$$u_y^{(2)} + 9\,u_x^{(1)} = 0$$

to canonical form and obtain its general solution and the particular solution such that, on $y = 0$, $\bar{u} = (2x, 3x)^T$

b) Reduce the system **[8]**

$$\bar{u}_y + \begin{pmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{pmatrix} \bar{u}_x = \bar{0}$$

to canonical form and hence find $\bar{u}$ such that $\bar{u} = (x, 2x, 3x)^T$ when $y = -x$.

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**Total No. of Questions : 8]** **[Total No. of Pages : 3**

# P812

## [3621] - 205

## M.A. / M.Sc.

## MATHEMATICS

## MT - 605 : Partial Differential Equations

## (New)

***Time : 3 Hours]*** ***[Max. Marks : 80***

***Instructions to the candidates:-***

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

***Q1)*** a) Explain the Charpit's method to find the solution of $f(x, y, z, p, q) = 0$. **[6]**

b) Find the general integral of the partial differential equation

$$(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$$

which contain the straight line $x(t) = 1, y(t) = 0, z(t) = t$. **[6]**

c) Obtain the partial differential equation by eliminating arbitrary constants from the eight circular cone $x^2 + y^2 = (z - c)^2 \tan\alpha$ where $\alpha$ and $c$ are arbitrary constants. **[4]**

***Q2)*** a) State and prove necessary and sufficient condition that the pfaffian differential is integrable. **[7]**

b) Show that the equation $z = px + qy$ is compatible with any equation $f(x, y, z, p, q) = 0$ which is homogeneous in $x, y, z$. **[5]**

c) Find a complete integral of the partial differential equation $z = px + qy + pq$. **[4]**

***Q3)*** a) Verify that the pfaffian differential equation

$$(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$$

is exact/integrable and find the corresponding integral. **[6]**

***P.T.O.***

b) Define compatible system of first order partial differential equations and give suitable example. **[2]**

c) Define characteristic strip. If an element $(x_0, y_0, z_0, p_0, q_0)$ is common to both an integral surface $z = z(x, y)$ and a characteristic strip, then show that the corresponding characteristic curve lies completely on the surface. **[8]**

***Q4)*** a) Explain the method of obtaining system of surfaces orthogonal to system of surfaces $f(x, y, z) = c$. **[5]**

b) Find a complete integral of the equation

$$p^2x + q^2y = z$$

by Jacobi method. **[5]**

c) Find the integral surface of the equation $pq = z$ passing through the curve $c : x_0 = 0,\ y_0 = t, z_0 = t^2$. **[6]**

***Q5)*** a) Obtain D' Alembert's solution for the initial value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$u(x, 0) = f(x),\ u_t(x, 0) = g(x),\ -\infty < x < \infty.$ **[8]**

b) State and prove maximum principle theorem. **[6]**

c) Find the Fourier transform of $f(x) = e^{-x^2}$. **[2]**

***Q6)*** a) Reduce the partial differential

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

to a canonical form and find its solution. **[6]**

b) State the problem of heat conduction of finite rod case and find its solution by separation of variables method. **[6]**

c) Show that the surfaces $x^2 + y^2 + z^2 = cx^{2/3}$ where $c$ is a constant, can form an equipotential family of surfaces, and find the general form of the potential function. **[4]**

*Q7)* a) State and prove convolution theorem for fourier transforms. **[6]**

b) State Neumann problem, obtain necessary condition for a problem to be a Neumann problem and show that the solution of the Neumann problem is unique up to the addition of a constant. **[10]**

*Q8)* a) Using Duhamel's principle find the solution of nonhomogeneous wave equation $u_{tt} - c^2 u_{xx} = f(x, t), -\infty < x < \infty, t > 0$

$u(x, 0) = u_t(x, 0) = 0, -\infty < x < \infty.$ **[6]**

b) If $\Phi = \Phi(r, \theta, \psi)$ is a harmonic function where $(r, \theta, \psi)$ are the spherical polar co-ordinate then show that

$$\tilde{\Phi} = \frac{a^2}{r}\,\Phi\left(\frac{a^2}{r}, \theta, \psi\right)$$

is also a harmonic function, where $a$ is a constant. **[6]**

c) State Cauchy problem with suitable example. **[4]**

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**Total No. of Questions : 8]** **[Total No. of Pages : 3**

# P814

## [3621] - 301
## M.A. / M.Sc.
## MATHEMATICS
## MT - 701 : Functional Analysis
## (New)

***Time : 3 Hours]*** ***[Max. Marks : 80***

***Instructions to the candidates:-***

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

***Q1)*** a) Let N be a non-zero normed linear space and prove that N is a Banach space if and only if $\{x \mid \|x\| = 1\}$ is complete. **[8]**

b) Consider the real linear space $R^2$ of all ordered pairs $x = (x_1, x_2)$ of real numbers. On $R^2$ define the norms

$$\|x\|_1 = |x_1| + |x_2|,$$

and

$$\|x\|_2 = (|x_1|^2 + |x_2|^2)^{1/2}.$$

Draw the closed unit spheres centered at origin with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$. **[4]**

c) Let $1 \le p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1.$

If $a$ and $b \ge 0$, then show that $a^{1/p}\, b^{1/q} \le \frac{a}{p} + \frac{b}{q}.$ **[4]**

***Q2)*** a) Let N and N' be normed linear spaces and T is a linear transformation of N into N'. Prove that T is continuous if and only if T is continuous at origin. **[6]**

b) Let T be an operator on a Banach space B. Show that T has an inverse $T^{-1}$ if and only if T* has inverse $(T^*)^{-1}$. **[4]**

c) If M is a closed linear subspace of a normed linear space N and $x_0$ is a vector not in M. Prove that there exists a functional $f_0$ in N* such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$. **[6]**

***P.T.O.***

*Q3)* a) If B and B' are Banach spaces and if T is a continuous linear transformation of B onto B'. Prove that the image of each open sphere centered on the origin in B contains an open sphere centred on the origin in B'. **[8]**

b) If N is a normed linear space, then prove that N* is a Housdorff space with respect to weak* topology. **[4]**

c) Prove that the dual space N* of a normed-linear space N is a Banach space. **[4]**

*Q4)* a) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm. **[6]**

b) If M is a closed linear subspace of a Hilbert space H, then show that $M = M^{\perp\perp}$. **[6]**

c) Let $X = R^2$. Find $M^{\perp}$ if $M = \{(x, y) \in R^2 \mid (x, y) \neq (0, 0)\}$. **[4]**

*Q5)* a) Let $\{e_1, e_2, .........., e_n\}$ be a finite orthonormal set in a Hilbert space H.

If $x$ is any vector in H, then prove that $\sum_{i=1}^{n} |(x, e_i)| \leq \|x\|^2$ ;

and $\left( x - \sum_{i=1}^{n} (x, e_i)\, e_i \right) \perp e_j$ for each $j \leq n$. **[6]**

b) Show that an orthonormal set in a Hilbert space is linearly independent and use this to prove that a Hilbert space is finite dimensional if and only if every complete orthonormal set is a basis. **[6]**

c) Let y be a fixed vector in a Hilbert space H and consider the function $f_y$ defined on H by $f_y(x) = (x, y)$. Show that $\| f_y \| = \| y \|$. **[4]**

*Q6)* a) For the adjoint operation $T \to T^*$ on (BCH) (H – a Hilbert space) prove that **[6]**

i) $T^{**} = T$,

ii) $\|T^*\| = \|T\|$,

iii) $\| T^*T \| = \| T \|^2$.

b) Let $A_1$ and $A_2$ be self-adjoint operators on a Hilbert space H. Prove that $A_1A_2$ is self-adjoint if and only if $A_1A_2 = A_2A_1$. **[4]**

c) Show that the unitary operators on a Hilbert space H form a group. **[6]**

***Q7)*** a) Prove that an operator T on a Hilbert space H is self-adjoint if and only if $(Tx, x)$ is real for all $x$. **[6]**

b) If P is a projection on a closed linear subspace M of a Hilbert space H, then prove that M is invariant under an operator T if and only if TP = PTP. **[6]**

c) Let P and Q be the projections on closed linear subspaces M and N of a Hilbert space H. If $P \leq Q$ then prove that Q-P is a projection. **[4]**

***Q8)*** a) Let T be a normal operator on a Hilbert space H with spectrum $\{\lambda_1, \lambda_2, \ldots\ldots, \lambda_m\}$ and use the spectral resolution of T to prove the following statements : **[8]**

i) T is self-adjoint if and only if each $\lambda_i$ is real.

ii) T is positive if and only if $\lambda_i \geq 0$ for each $i$.

b) If T is a normal operator on a Hilbert space H, then prove that $x$ is an eigenvector of T with eigenvalue $\lambda$ if and only if $x$ is an eigenvector of T* with eigenvalue $\overline{\lambda}$. **[4]**

c) Find the spectrum of the operator $T : R^3 \to R^3$ defined by

$T(1, 0, 0) = (0, 1, 0), T(0, 1, 0) = (0, 0, 1), T(0, 0, 1) = (1, 0, 0)$. **[4]**

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**P814**

# [3621] - 301
## M.A. / M.Sc.
## MATHEMATICS
## MT - 701 : General Topology
## (Old)

*Time : 3 Hours]*	*[Max. Marks : 80*

***Instructions to the candidates:-***

1) ***Answer any five questions.***
2) ***Figures to the right indicate full marks.***

***Q1)*** a) Show that the topologies of $R_l$ and $R_k$ are strictly finer than the standard topology on R. **[6]**

b) Show that the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps. Further, show that

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ is open in } X\} \cup \{\pi_2^{-1}(V)/V \text{ is open in } Y\}$$

is a subbasis for the product topology on X × Y. **[5]**

c) In the finite complement topology on R, to what point or points does the sequence $x_n = 1/n$ converge. **[5]**

***Q2)*** a) Let Y be a subspace of X and let A be a subset of Y. Let $\overline{A}$ denote the closure of A in X. Show that the closure of A in Y equals in $\overline{A} \cap Y$. **[6]**

b) Show that X is Hausdorff if and only if $\Delta = \{x \times x / x \in X\}$ is closed in X×X. **[5]**

c) Find the boundary and the interior of each of the following subsets of $R^2$. **[5]**

i) $A = \{x \times y / y = 0\}$

ii) $B = \{x \times y / x \text{ is rational}\}$.

*P.T.O.*

*Q3)* a) Let Y be an ordered set in the order topology. Let f,g : X $\rightarrow$ Y be continuous.

i) Show that the set $\{x / f(x) \leq g(x)\}$ is closed in X.

ii) Let $h : X \rightarrow Y$ be the function

$$h(x) = \min \{f(x), g(x)\}.$$

Show that $h$ is continuous. **[6]**

b) State and prove the Pasting Lemma. **[5]**

c) Let $f : R \rightarrow R^w$ be defined by **[5]**

$$f(t) = (t, t, t, .......).$$

show that $f$ is not continuous if $R^w$ is given the box topology.

*Q4)* a) Show that the topologies on $R^n$ induced by the euclidean metric $d$ and the square metric $\rho$ are the same as the product topology on $R^n$. **[6]**

b) Let $fn : X \rightarrow Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If $(fn)$ converges uniformly to $f$, then show that $f$ is continuous. **[5]**

c) Let $\pi_1 : R \times R \rightarrow R$ be projection on the first coordinate. Let A be the subspace of $R \times R$ consisting of all points $x \times y$ for which either $x \geq 0$ or $y = 0$ (or both). Let $q : A \rightarrow R$ be obtained by restricting $\pi_1$. Show that $q$ is a quotient map that is neither open nor closed. **[5]**

*Q5)* a) Show that the image of a connected space under a continuous map is connected. Hence show that every continuous function $f : [0,1] \rightarrow Z$ is constant. **[6]**

b) Show that if U is an open connected subspace of $R^2$, then U is path connected. **[5]**

c) If X is locally path connected, then show that the components and the path components of X are the same. **[5]**

*Q6)* a) Show that every closed subspace of a compact space is compact. Further, show that compact subspace of a topological space need not be closed. **[6]**

b) Show that a connected metric space having more than one point is uncountable. **[5]**

c) Show that [0, 1] is not limit point compact as a subspace of $R_l$. **[5]**

*Q7)* a) Show that the space $R_k$ is Hausdorff but not regular. **[6]**

b) Show that every metrizable Lindelöf space has a countable basis. **[5]**

c) Show that the rationals Q are not locally compact. **[5]**

*Q8)* a) Show that every regular space with countable basis is normal. **[6]**

b) Show that a subspace of completely regular space is completely regular. **[5]**

c) State and prove Urysohn lemma for a metric space (X, d). **[5]**

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Total No. of Questions : 8] [Total No. of Pages :2

**P815**

# [3621] - 302

# M.A. / M.Sc. (New)

# MATHEMATICS

## MT - 702 : Ring Theory

*Time : 3 Hours]* *[Max. Marks :80*

***Instructions to the candidates:***

1) ***Answer any five questions.***
2) ***Figures to the right indicate full marks.***

***Q1)*** a) Let R be an integral domain and let R [$x$] be the ring of polynomials over R. Then prove that the units of R [$x$] are just the units of R and also prove that R [$x$] is an integral domain. **[6]**

b) Let R be a commutative ring. Then prove that the ideal P in R is prime if and only if the quotient ring R/P is an integral domain. **[6]**

c) Prove that a field has only two idempotents. **[4]**

***Q2)*** a) Prove the ring Z [$x$] of polynomials over integers is a factorization domain but not principle ideal domain. **[6]**

b) Let $x$ be a nilpotent element of the commutative ring R then prove that

i) $x$ is either zero or a zero divisor

ii) $r\,x$ is nilpotent for all $r \in$ R

iii) $1 + x$ is unit in R. **[6]**

c) Give an example of a commutative ring with unity of characteristic two which is not a Boolean ring. Justify your answer. **[4]**

***Q3)*** a) Prove that if F is a field, then the polynomial ring F[$x$] is an Euclidean domain. **[6]**

b) Prove that in the ring of continuous functions C (R), the ideal.
$I = \{f \in C(R) \setminus f(1/2) = 0\}$ is maximal. **[6]**

c) Prove that the ring Z[$x$] and Q[$x$] are not isomorphic. **[4]**

***Q4)*** a) State and prove the Eisenstein's Criterion. **[6]**

b) Prove that the quotient of a UFD need not be a UFD. **[6]**

*P.T.O.*

c) Let f : R $\rightarrow$ S be homomorphism of rings. Prove that if S is an integral domain then kernel (f) is prime ideal. **[4]**

***Q5)*** a) Let R be a commutative ring with unity. Then prove that R has exactly one prime ideal if and only if every element of R is nilpotent or a unit. **[6]**

b) If R is a Noetherian ring then prove that the polynomial ring R[$x$] is also Noetherian ring **[8]**

c) Define affine algebraic set. **[2]**

***Q6)*** a) If R is a Noetherian ring then prove that for any ideal I, some positive power of rad (I) is contained in I. **[6]**

b) Prove that an Artinian Integral domain is a field. **[6]**

c) Prove that R is a Dedekind Domain then R is a PID. if and only if R is a UFD. **[4]**

***Q7)*** a) If R is Noetherian, integrally closed integral domain with unique non zero prime ideal and also R is local ring then prove that R is a Noetherian integral domain that is also a local ring whose unique maximal ideal is non zero and principal. **[8]**

b) State and prove the Gauss Lemma for irreducibility of polynomials. **[8]**

***Q8)*** a) Prove that the polynomial $f(x) = x^{P-1} + x^{P-2} + .......+ x + 1$ is irreducible over Q. **[6]**

b) Prove that every Artinian ring is Noetherian ring. **[6]**

c) Prove that the ring of integers in an algebraic number field is a Dedekind domain. **[4]**

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**P815**

# [3621] - 302
# M.A. / M.Sc. (Old)
# MATHEMATICS
## MT - 702 : Mechanics

*Time : 3 Hours]* *[Max. Marks :80*

***Instructions to the candidates:***

1) ***Answer any five questions.***
2) ***Each question carries 16 marks.***

***Q1)*** a) Explain the following terms.

i) Generalised coordinates ii) Principle of virtual work. **[4]**

b) Discuss the limitations of Newtonian mechanics. **[4]**

c) Derive Lagrange's equations of motion using the De Alembert's principle. **[8]**

***Q2)*** a) Define a rigid body. Show that the constraint of rigidity is conservative in nature. **[6]**

b) Show that if the Hamiltonian does not explicitly depend on time then it is a constant of motion. **[5]**

c) For the following Lagrangian find the corresponding Hamiltonian :

$L(x, \dot{x}) = \frac{1}{2}(\dot{x})^2 - \frac{1}{2}w^2x^2 + \alpha x\,(\dot{x})^2$, where $w$ and $\alpha$ are constants. **[5]**

***Q3)*** a) Show that Poisson bracket is a bilinear operation. **[2]**

b) Prove the Jacobi identity. **[7]**

c) For a simple pendulum of length 1, the Lagrangian is

$$L = m\left(l^2\dot{\theta}^2 - gl\theta^2\right)$$

where m, l and g are constants. Show that the system traces an ellipse in the phase plane. **[7]**

***Q4)*** a) Find the stationary function of the integral

$$\int_0^4 [x\,\dot{y} - \dot{y}^2]\,dx,$$

where $\dot{y} = \frac{dy}{dx}$, and y(0) = 0, y(4) = 3. **[5]**

***P.T.O.***

b) Write the Lagrangian for a projectile moving under the force of gravity. Find its Hamiltonian and write the Hamilton's equations of motion. **[5]**

c) A particle is constrained to remain on the plane $x + y + z = 0$ but is otherwise free. Choose $x$, $y$ as generalized co-ordinates and write down the Lagrangian of the particle, and hence obtain its generalized momentum $p_x$. **[6]**

***Q5)*** a) Derive the symplectic condition for a transformation to be canonical. **[8]**

b) Show that the following transformation is canonical.

$$Q_1 = q_1,\ P_1 = p_1 - 2p_2,\ Q_2 = p_2,\ P_2 = -2q_1 - q_2.$$ **[8]**

***Q6)*** a) State and prove Liouville's theorem. **[5]**

b) Define Poisson brackets and show that they are invariant under restricted canonical transformations. **[6]**

c) If u, v are two dynamical variables, then show that [u, v] is a constant of motion, if u and v are constants of motion. **[5]**

***Q7)*** a) Find a generating function for $Q = q^{1/2} \cos(2p)$, $P = q^{1/2}\sin(2p)$. **[8]**

b) Write Hamilton-Jacobi equation for 1dimensional simple harmonic oscillator and solve it. **[8]**

***Q8)*** a) A particle moves along the curve,

$$\bar{r} = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k},$$

starting at t = 0. Find velocity and acceleration at $t = \pi/2$. **[5]**

b) Define central force motion. Show that it is always planar. Further show that the areal velocity in this case is directly proportional to time. **[5]**

c) In case of three body motion, show that the center of motion of the 3 bodies either remains at rest or moves uniformly on a straight line. **[6]**

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**P816**

# [3621] - 303
# M.A. / M.Sc.
# MATHEMATICS
## MT - 703 : Mechanics (New)

*Time : 3 Hours]* *[Max. Marks :80*

***Instructions to the candidates:***

1) ***Answer any five questions.***
2) ***Each question carries 16 marks.***

***Q1)*** a) Explain the following :

i) Principle of virtual work  ii) D' Alembert's principle,

iii) Generalised coordinates,  iv) Degrees of freedom. **[8]**

b) Classify the constraints in four types. Give examples of each type. **[8]**

***Q2)*** a) Derive Lagrange's equations of motion using the Hamilton's principle.**[8]**

b) Write the Lagrangian for Atwood's machine and hence write the Lagrange's equations of motion. **[5]**

c) Define generalised momentum and show that in absence of non potential forces, the momentum conjugate to a cyclic co-ordinate is conserved.**[3]**

***Q3)*** a) If the Hamiltonian H of the system is

$$H(q_1, q_2, p_1, p_2) = q_1p_1 - q_2p_2 - aq_1^2 - bq_2^2.$$

then show that $q_1q_2$ and $\dfrac{p_1 - aq_1}{q_2}$ are constants of motion, where $a$, $b$ are constants. **[6]**

b) Write the Lagrangian and Hamiltonian for a projectile moving under earth's constant field of gravity. Which are the cyclic co-ordinates? Which are the conserved quantities? **[5]**

c) Show the following transformation is canonical.

$Q = q\cos\alpha - p\sin\alpha$, $P = q\sin\alpha + p\cos\alpha$, where $\alpha$ is a constant. **[5]**

***Q4)*** a) Explain the brachistocrone problem. **[5]**

***P.T.O.***

b) Find the extremal curve of the given functional :

$$\int_0^{\pi}\left(y'^2 - y^2 + 4y\cos x\right)dx,\ y(0) = 0,\ y(\pi) = 0.$$ **[5]**

c) Show that

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right) = \frac{\partial \dot{x}_i}{\partial q_j},$$

where $x_i = x_i\,(q_j, t)$, where $i, j = 1, \ldots, n$. **[6]**

***Q5)*** a) Derive the symplectic condition for a transformation to be canonical. **[6]**

b) Prove the Jacobi identity for any three dynamical variables $u, v, w$. **[6]**

c) Show that Poisson brackets satisfy $[UV, W] = [U, W]V + U[V, W]$. **[4]**

***Q6)*** a) State and prove the Euler's theorem for rigid body motion. **[5]**

b) Prove the rotation formula:

$r' = r \cos \Phi + n(n.r)\,(1 - \cos \Phi) + (r \times n) \sin \Phi.$ **[6]**

c) Define infinitesimal rotations. Show that infinitesimal rotations commute. **[5]**

***Q7)*** a) Find the transformation generated by

$F_1\,(q, Q) = qQ - mwq^2/2 - Q^2/\,(4mw),$

where $m, w$ are constants. **[6]**

b) Explain the concept of Euler angles. **[5]**

c) Show that if Hamiltonian H of a system does not depend on time explicitly then it represents total energy of the system. **[5]**

***Q8)*** a) Show that if the law of central force is an inverse square law of attraction, then the path of the particle is a conic. **[8]**

b) Define central force motion. Show that it is always planar. Further show that the areal velocity in this case is constant. **[4]**

c) Show that the angular momentum and total energy are conserved in a central force motion. **[4]**

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**Total No. of Questions : 8]** **[Total No. of Pages :2**

# P816

# [3621] - 303
# M.A. / M.Sc.
# MATHEMATICS
## MT - 703 : Functional Analysis (Old)

***Time : 3 Hours]*** ***[Max. Marks :80***

***Instructions to the candidates:***

1) ***Answer any five questions.***
2) ***Figures to the right indicate full marks.***

***Q1)*** a) Define a Banach space. Let N = (0, 1), on N.
Define the function $\|\cdot\|$ as follows :
for $x \in$ N, $\|x\| = |x|$.
i) Is N a normed linear space.
ii) If so, is N a Banach space? Justify. **[6]**

b) State and prove Minkowski's inequality. **[6]**

c) Show that T : $l_2 \rightarrow l_2$ defined by T$(x_1, x_2,....)$ = $(0, x_1, x_2,....)$ is a continuous linear operator on $l_2$ and find its norm $\|$T$\|$. **[4]**

***Q2)*** a) State and prove the Hahn - Banach theorem. **[6]**

b) Discuss the natural imbedding of N in N**. **[8]**

c) Define reflexive space. **[2]**

***Q3)*** a) State and prove open mapping theorem. **[6]**

b) State and prove the uniform boundedness theorem. **[6]**

c) Let B be a Banach space and N a normed linear space. If $\{T_n\}$ is a sequence is B(B,N) such that T$(x)$ = $\lim_{n\to\infty} T_n(x)$ exists for each $x$ in B, prove that T is a continuous linear transformation. **[4]**

***Q4)*** a) If $x$ and $y$ are any two vectors in a Hilbert space, then prove that
$|(x, y)| \leq \|x\| \, \|y\|$. **[6]**

b) Let M be a closed linear subspace of a Hilbert space H, let $x$ be a vector not in M, and let $d$ be the distance from $x$ to M.
Prove that there exists a unique vector $y_0$ in M such that $\|x - y_0\| = d$. **[6]**

c) Show that the parallelogram law is not true in $l_1^n$ $(n > 1)$. **[4]**

***P.T.O.***

*Q5)* a) State and prove Bessel's inequality. **[8]**

b) Let H be a Hilbert space, and let $f$ be an arbitrary functional in H*. Prove that there exists a unique vector $y$ in H such that

$$f(x) = (x, y)$$

for every $x$ in H. **[8]**

*Q6)* a) Let T be an operator on a Hilbert space H for which $(Tx, x) = 0$ for all $x$. Prove that T = 0. **[6]**

b) Prove that an operator T on a Hilbert space H is normal if and only if $\|T^*x\| = \|Tx\|$ for every $x$ in H. **[6]**

c) Show that the adjoint operation is one-to-one and onto as a mapping of B(H) into itself. **[4]**

*Q7)* a) Let P be a projection on a Hilbert space H with range M and null space N. Prove that $M \perp N$ if and only if P is self adjoint, and in this case $N = M^{\perp}$. **[8]**

b) Prove that a closed linear subspace M of a Hilbert space H is invariant under an operator T if and only if $M^{\perp}$ is invariant under T*. **[6]**

c) If P is a projection on a closed linear subspace M of a Hilbert space H, then prove that

$$\|Px\| = \|x\|.$$ **[2]**

*Q8)* a) If T is a operator on a Hilbert space H, define eigenvector of T, eigenvalue of T, eigenspace of T. Show that the eigenspace is a subspace of H. **[6]**

b) Show that an operator T on a finite dimensional Hilbert space H is normal if and only if its adjoint T* is a polynomial in T. **[6]**

c) Find the spectrum of the operator

$T : R^2 \to R^2$ defined by

$T(1, 0) = (0, -1)$ and $T(0, 1) = (-1, 0)$. **[4]**

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**Total No. of Questions : 8]** **[Total No. of Pages :3**

**P817**

# [3621] - 304

# M.A. / M.Sc. (New)

# MATHEMATICS

## MT - 704 : Measure and Integration

***Time : 3 Hours]*** ***[Max. Marks :80***

***Instructions to the candidates:***

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***
3) ***(X, B, $\mu$) is a measure space.***

***Q1)*** a) If (X, B, $\mu$) is a measure space prove that there exists a measure space (X, $B_0$, $\mu_0$) such that

i) $B \subseteq B_0$

ii) $E \in B \Rightarrow \mu(E) = \mu_0(E)$.

iii) $E \in B_0 \Leftrightarrow E = A \cup B$, with $B \in B$, and $A \subseteq C$, with $C \in B$ and $\mu(C) = 0$. **[8]**

b) Let $f : X \to R$ be a non negative measurable function. Prove that there is a sequence $\varphi_n$ of simple functions with $\varphi_{n+1} \geq \varphi_n$ such that $f = \lim \varphi_n$ at each point of X. **[8]**

***Q2)*** a) If $f : X \to R$ is nonnegative and measurable, define $\int_X f\, d\mu$. **[4]**

b) If X = N , B = all subsets of N, and $\mu$ = counting measure, and $f : X \to R$ is $f(n) = 2^{-n}$, Evaluate $\int_N f\, d\mu$. **[4]**

c) Let $f_n : X \to R$ be a sequence of nonnegative measurable functions that converge on X to $f$. Prove that $\int_X f\, d\mu \leq \underline{\lim} \int_E f_n\, d\mu$. **[8]**

***Q3)*** a) Let $f : X \to R$ be integrable. Show that given $\varepsilon > 0$ there is a $\delta > 0$ such that for each measurable set E with $m(E) < \delta$, we have $\left|\int_E f\right| < \varepsilon$. **[4]**

***P.T.O.***

b) i) Define a signed measure on a measure space. **[2]**

ii) If $v$ is a signed measure on (X, B) and $E \subseteq X$ is such that $0 < v(E) < \infty$, prove that there is a positive set $A \subseteq E$ with $v(A) > 0$. **[4]**

c) Give an example of a sigma algebra on R with four elements. Define a measure on this space. **[6]**

***Q4)*** a) Prove the Hahn decomposition theorem. **[8]**

b) Give an example to show that the Hahn decomposition need not be unique. **[4]**

c) Give an example of a signed measure on a measure space, which has a positive set and a negative set. **[4]**

***Q5)*** a) Let μ and $v$ be measures on a space (X, B).

i) When are μ and $v$ said to be mutually singular? Give an example. **[2]**

ii) When is $v$ said to be absolutely continuous with respect to μ? Give an example. **[2]**

b) Prove the Radon - Nikodym theorem. **[6]**

c) If $f \in L^2(X, B, \mu)$, prove that given $\varepsilon > 0$, there is a simple function φ vanishing outside a set of finite measure, such that $\| f - \varphi \|_2 < \varepsilon$. **[6]**

***Q6)*** a) Let μ* be an outer measure on a space X. Prove that

i) The class B of $\mu^*$ measurable sets is a sigma algebra, and **[4]**

ii) If $\bar{\mu}$ is $\mu^*$ restricted to B, then $\bar{\mu}$ is a complete measure on B. **[4]**

b) If $\mu$ is a finite Baire measure on R, prove that its cumulative distribution function F is a monotopic increasing bounded function which is continuous from the right. **[8]**

***Q7)*** a) State Fubini's theorem. **[4]**

b) If $f, g \in L'(R)$, define $h(x) = f * g(x) = \int_R f(x-t)\, g(t)\, dt$.

i) Prove that $h \in L'(R)$. **[4]**

ii) If $f_1, f_2$ and $f_3 \in L'(R)$, prove that $(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$. **[4]**

c) Calculate $\chi_{[0,1]} * \chi_{[0,1]}$. **[4]**

***Q8)*** a) What is a Caratheodory outer measure? **[4]**

b) State the Riesz-Markov theorem. **[4]**

c) Give an example of a positive linear functional on $C_c(R)$. **[4]**

d) Give an example of an invariant measure on the groups.

i) (R, +) **[2]**

ii) (Z, +). **[2]**

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**P817**

# [3621] - 304

# M.A. / M.Sc. (Old)

# MATHEMATICS

## MT - 704 : Mathematical Methods - I

***Time : 3 Hours]*** ***[Max. Marks :80***

***Instructions to the candidates:***

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

***Q1)*** a) Test for convergence the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+.........+\frac{1}{n}+.........$$ **[4]**

b) Test for convergence $\sum_{n=2}^{\infty}\frac{3^n-n^3}{n^5-5n^2}$. **[4]**

c) Use division of two series, to find the series for

i) $\frac{1}{x}\log(1+x)$ ii) $\tan x$. **[8]**

***Q2)*** a) Evaluate by using L' Hopital's rule $\lim_{x\to 1}\frac{\log(2-x)}{x-1}$. **[4]**

b) Find the Fourier sine series and Fourier cosine series which represents $f(x) = (\pi - x)$ in $0 < x < \pi$. **[8]**

c) Prove that $F\{g(kx)\} = \frac{1}{k}G\left(\frac{s}{k}\right)$. **[4]**

***Q3)*** a) Define the Dirac delta function, $\delta(x)$. **[4]**

b) Prove that if the functions g($x$) and f(k) are absolutely integrable on $(-\infty,+\infty)$ and that the Fourier inversion integral for $f(x)$ is valid for all $x$ except possibly at a countably infinite number of points, then

$$\int_{-\infty}^{\infty} F(k)G(-k)\,dk = \int_{-\infty}^{\infty} f(x)\,g(x)\,dx,$$

where $F(k) = \phi\{f(x)\}$, $G(k) = \phi\{g(x)\}$. **[12]**

***P.T.O.***

*Q4)* a) Obtain Maclaurin's series for $\sin^{-1}x$. **[8]**

b) Prove that associated Legendre functions

$$P_l^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x).$$ **[8]**

*Q5)* a) Establish the integral formula for Bessel function $J_n(x)$ of the first kind in the form $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta$. **[8]**

b) Prove that

i) $H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}$. **[4]**

ii) $H_{2n+1}(0) = 0$. **[4]**

*Q6)* a) State and prove Rodrigue's formula for Legendre's equation. **[8]**

b) Prove that $L_n(0) = 1$. **[4]**

c) Evaluate $\int_0^{\infty} \sqrt{x}.e^{-x^3}.dx$. **[4]**

*Q7)* a) Compute $\Gamma(-1/2)$. **[4]**

b) State and prove the first and second shifting properties of Laplace transform. **[8]**

c) Find $L\{t\}$, by using Laplace transform of derivative of a function. **[4]**

*Q8)* a) Obtain $L^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\}$. **[8]**

b) Find the solution to the initial value problem $y''(t) + 4y'(t) + 8y(t) = \sin t$, where $y(0) = 1$ and $y'(0) = 0$. **[8]**

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Total No. of Questions : 8] [Total No. of Pages :3

**P818**

# [3621] - 305
# M.A. / M.Sc. (New)
# MATHEMATICS
## MT - 705 : Graph Theory

*Time : 3 Hours]* *[Max. Marks :80*

***Instructions to the candidates:***

1) ***Answer any five questions.***
2) ***Figures to the right indicate marks.***

***Q1)*** a) Prove that if a graph has no odd cycles then it is bipartite. **[6]**

b) How many simple graphs are there on a fixed set of four vertices? Draw all non isomorphic simple graphs with four vertices. **[6]**

c) Determine whether the Petersen graph is bipartite, and find the size of its largest independent set. **[4]**

***Q2)*** a) Prove that if a graph has at most one nontrivial component and its vertices all have even degree then it is Eulerian. **[6]**

b) Determine the values of $m$ and $n$ such that $K_{m,n}$ is Eulerian. **[4]**

c) Determine whether the sequence (5, 5, 4, 3, 2, 2, 2, 1) is graphic. Provide a construction or a proof of impossibility. **[6]**

***Q3)*** a) Prove that the center of a tree is a vertex or an edge. **[6]**

b) Compute the diameter and radius of each of the following graphs i) $K_{m,n}$ ii) the cycle $C_n$. **[6]**

c) Let T be a tree with average degree $a$. In terms of $a$, determine $n(T)$. **[4]**

***Q4)*** a) Prove that if S is a $n$-element subset of the set of natural numbers N then there are $n^{n-2}$ trees with vertex set S. **[6]**

b) Determine which trees have Prüfer codes that

i) contain only one value.

ii) have distinct values in all positions. **[4]**

c) There are five cities in a network. The cost of building a road directly between $i$ and $j$ is the entry $a_{ij}$ in the matrix below. An infinite entry indicates that there is mountain in the way and the road cannot be built.

***P.T.O.***

Determine the least cost of making all the cities reachable from each other.

$$\begin{bmatrix} 0 & 3 & 5 & 11 & 9 \\ 3 & 0 & 3 & 9 & 8 \\ 5 & 3 & 0 & \infty & 10 \\ 11 & 9 & \infty & 0 & 7 \\ 9 & 8 & 10 & 7 & 0 \end{bmatrix}.$$ **[6]**

***Q5)*** a) Prove that if G is an X, Y – bigraph and $|N(S)| \geq |S|$ for all $S \subseteq X$ then G has a matching that saturates X. **[6]**

b) i) Determine the minimum size of a maximal matching in the cycle $C_n$.

ii) Find a maximum matching in the graph $K_{4,4}$. **[6]**

c) Let T be a tree with $n$ vertices, and let $k$ be the maximum size of an independent set in T. Determine $\alpha'(T)$ in terms of $n$ and $k$. **[4]**

***Q6)*** a) Prove that if G is a bipartite graph then the maximum size of a matching in G equals the minimum size of a vertex cover of G. **[6]**

b) i) Illustrate the statement in part (a) by an example.

ii) Prove or disprove : Every tree has at most one perfect matching. **[6]**

c) Using non negative edge weights, construct a 4 - vertex weighted graph in which the matching of maximum weight is not a matching of maximum size. **[4]**

***Q7)*** a) Prove that if G is a simple graph then $K(G) \leq K'(G) \leq \delta(G)$. Give an example of a graph where inequality holds strictly. **[6]**

b) Prove that if a graph has an ear decomposition then it is 2-connected. **[4]**

c) Prove or disprove each statement below.

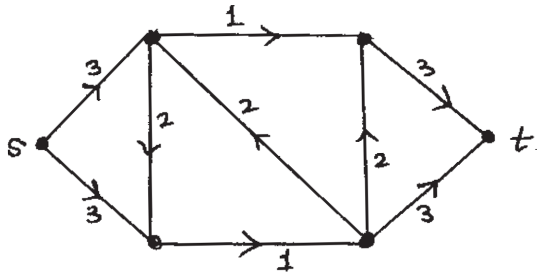
i) Every graph with connectivity 4 is 2-connected.

ii) Every 3-connected graph has connectivity 3. **[6]**

***Q8)*** a) Prove that if $x$ and $y$ are distinct vertices of a graph G then the minimum size of an $x, y$ – disconnecting set of edges equals the maximum number of pair wise edge-disjoint $x, y$ – paths. Illustrate with an example. **[8]**

b) In the network below, list all integer-value of feasible flows and select a flow of maximum value. Prove that this flow is a maximum flow by exhibiting a cut with the same value.

Determine the number of source / sink cuts.




**[8]**

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Total No. of Questions : 8] [Total No. of Pages :2

**P818**

# [3621] - 305

# M.A. / M.Sc. (Old)

# MATHEMATICS

## MT - 705 : Rings and Modules

*Time : 3 Hours]* *[Max. Marks :80*

***Instructions to the candidates:***

1) ***Answer any five questions.***

2) ***Figures to the right indicate marks.***

***Q1)*** a) Prove that if R = C (X, R) then

i) R has no non-trivial nilpotent element and

ii) R has no non-trivial idempotents if and only if X is connected. **[6]**

b) Show that the ring R = C([0, 1], R) is not an integral domain. **[5]**

c) Show that the subset Z [$i$] = $\{a + ib \mid a, b \in Z\}$ of C is a subring of C. **[5]**

***Q2)*** a) Prove that the characteristic of an integral domain is either $o$ or $a$ prime number. **[6]**

b) Show that the inter section of any family of subrings of a ring is a subring. **[5]**

c) Give examples of two subrings of Z whose union is not a subring. **[5]**

***Q3)*** a) Prove that if R is a ring with 1 and I is a ideal in R such that I ≠ R, then there is a maximal ideal M of the same kind as I such that I ⊆ M. **[6]**

b) Show that the group (Q, +) has no maximal subgroups. **[5]**

c) Give an example of a ring in which an ideal of an ideal is not an ideal. **[5]**

***Q4)*** a) Let R be a commutative ring with 1 and I be an ideal in R. Prove that I is a prime ideal if and only if $\frac{R}{I}$ is an integral domain. **[6]**

b) Show that a maximal ideal in a commutative ring with unity is a prime ideal but not conversely. **[5]**

c) In a commutative ring, show that the set of all nilpotent elements is an ideal. **[5]**

***P.T.O.***

*Q5)* a) Let I be an ideal in a ring R. Prove that I is a 2-sided ideal in R if and only if I is the Kernel of some homomorphism $f: R \rightarrow S$ for a suitable ring S. **[6]**

b) Prove that a homomorphism $f: R \rightarrow S$ between rings is a monomorphism if and only if ker $(f) = (0)$. **[5]**

c) Give an example to show that an epimorphic image of a prime ideal need not be one such. **[5]**

*Q6)* a) Prove that (Q (R), + , .) is a field containing R as a subring. **[6]**

b) Show that Q (Z) = Q (2Z). **[5]**

c) Is it true that the inverse image of a prime ideal (under homomorphism) is a prime ideal? Justify. **[5]**

*Q7)* a) Prove that every Euclidean domain is a PID (with 1). **[6]**

b) Show that the ring of integers Z is Euclidean. **[5]**

c) Prove that if R is a domain then R[X] is a UFD. **[5]**

*Q8)* a) Prove that a vector space is a free module. **[6]**

b) Give an example of a non-free module. **[5]**

c) Give an example of a free module in which a linearly independent subset cannot be extended to a basis. **[5]**

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Total No. of Questions : 8] [Total No. of Pages : 2

**P821**

# [3621] - 401
# M.A. / M.Sc.
# MATHEMATICS
# MT - 801 : Algebraic Topology

*Time : 3 Hours]* *[Max. Marks : 80*

***Instructions to the candidates:-***

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

***Q1)*** a) If $x_0, x_1 \in$ X, and $f$ is a path in X from $x_0$ to $x_1$, and $\tilde{f}(s) = f(1 - s)$, prove that $f * \tilde{f}$ is homotopic to the constant path at $x_0$. **[6]**

b) Let [X, [0, 1]] denote the set of homotopy classes of maps from X into [0, 1]. Prove that this set has a single element. **[4]**

c) Let $A \subseteq X$, and $r : X \to A$ be such that $r(a) = a, \forall\ a \in A$. If $a_0 \in A$, prove that $r_* : \pi_1(X, x_0) \to \pi_1(A, a_0)$ is surjective. **[6]**

***Q2)*** a) Define the fundamental group $\pi_1(X, x_0)$. **[4]**

b) Give an example of a connected space X for which **[4]**

i) $\pi_1(X, x_0) = \{e\}$.

ii) $\pi_1(X, x_0) = Z$. Justify your answer.

c) Let $p : E \to B$ be a covering map, and $p(e_0) = b_0$. Prove that any path $f : [0, 1] \to B$ beginning at $b_0$ has a unique lifting to a path $\tilde{f}$ in E beginning at $e_0$. **[6]**

d) Give a covering space of $S^1$. Justify your answer. **[2]**

***Q3)*** a) Define a covering space. **[2]**

b) Let $p : R \to S'$ be $p(t) = \exp.(2\pi it)$
and $f : [0, 1] \to S^1$ be $f(t) = \exp.(6\pi it)$
Find a lift of $f$ to R starting at 4. **[4]**

c) Prove that there is no retraction of $B^2$ onto S'. **[4]**

d) Prove that the fundamental group of the torus is isomorphic to Z × Z. **[6]**

*P.T.O.*

*Q4)* a) If $f: [0, 1] \to [0, 1]$ is continuous, prove that $\exists\ x_0 \in [0, 1]$ such that $f(x_0) = x_0$. **[4]**

b) If $f: B^2 \to B^2$ is continuous, prove that $\exists\ x_0 \in B^2$ such that $f(x_0) = x_0$. **[6]**

c) i) Define the real projective plane $P^2$. **[2]**

ii) Find the fundamental group of $P^2$. **[4]**

*Q5)* a) Let $f(x) = x^n + \sum_{k=0}^{n-1} a_k x^k$, with $\sum_{k=0}^{n-1} |a_k| < 1$. If $f$ has no root in the unit ball $B^2$, prove that $f$ is homotopic to the map $g(x) = x^n$. **[6]**

b) Find a circle around the origin containing all the roots of the equation $f(x) = x^7 + x^2 + 1 = 0$. **[6]**

c) Let $X = R^3$, $B$ = z-axis in $R^3$, and $Y = R^3/B$. Prove that Y has the punctured $xy$ plane $(R^2 \setminus [03] \times \{0\})$ as a deformation retract. **[4]**

*Q6)* a) Let $f: X \to Y$, $f(x_0) = y_0$.

i) When is $f$ a homotopy equivalence? **[4]**

ii) If $f$ is a homotopy equivalence, prove that $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism. **[4]**

b) If A is a deformation retract of X and B is a deformation retract of A, prove that B is a deformation retract of X. **[4]**

c) Find the fundamental group of $S' \cup (R_+ \times \{0\})$. **[4]**

*Q7)* a) Prove that if $n \geq 2$, $S^n$ is simply connected. **[6]**

b) Give examples of a curve C in the torus **[6]**

i) that does not separate the torus.

ii) that does separate the torus.

c) State the Jordan curve theorem. Define all the terms that you use. **[4]**

*Q8)* a) State the Jordan separation theorem. Define all terms that you use. **[4]**

b) i) Define the first homology group of a space X; $H_1(x)$. **[6]**

ii) Calculate $H_1(s^1)$.

c) When are two covering spaces of a space B said to be equivalent? Prove that this relation is an equivalence relation. **[6]**

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**Total No. of Questions : 8]** **[Total No. of Pages : 2**

# P822

## [3621] - 402
## M.A. / M.Sc.
## MATHEMATICS
## MT - 802 : Hydrodynamics

***Time : 3 Hours]*** ***[Max. Marks : 80***

***Instructions to the candidates:-***

1) ***Answer any five questions.***
2) ***Figures to the right indicate full marks.***

***Q1)*** a) Explain the terms : Streamline, pathline and velocity potential. Are the streamlines and pathlines of particles of a fluid always the same? Give reasons. **[6]**

b) Derive the general equation of continuity which must hold at any points of a fluid free from sources and sinks. **[8]**

c) Find the streamlines if $u = -wy,\ v = wx,\ w = 0$, where $[u,v,w]$ are the cartesian components of the particle velocity $\bar{q}$ and $w$ is a constant.**[2]**

***Q2)*** a) Obtain the condition that the surface $F(x, y, z, t) = 0$ may be a boundary surface, when the boundary is in motion. **[8]**

b) Show that the variable ellipsoid $\frac{x^2}{a^2k^2t^4} + kt^2\left[\left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2\right] = 1$ ($k$ = const.) is a possible form for the boundary surface of a liquid at any time $t$.**[8]**

***Q3)*** a) Find the equation of motion in usual notations in the form $\frac{d\bar{q}}{dt} = \bar{F} - \frac{1}{\rho}\nabla p$ and derive its cartesian equivalence. **[9]**

b) Test whether the motion specified by $\bar{q} = \frac{k^2(x\bar{j} - y\bar{i})}{x^2 + y^2}$ ($k$ = const.) is of the potential kind and if so determine the velocity potential. **[7]**

*P.T.O.*

*Q4)* a) State and prove the theorem of Blasius. **[8]**

b) A source of fluid situated in space of two dimensions is of such strength that $2\pi\rho\mu$ represents the mass of the fluid of density ρ emitted per unit time. Show that the force necessary to hold a circular disc at rest in the plane of the source is $\dfrac{2\pi\rho\mu^2 a^2}{r(r^2 - a^2)}$,where *a* is the radius of the disc and *r* the distance of the source from its centre. In what direction is the disc urged by the pressure? **[8]**

*Q5)* a) Explain a line source and a line sink in two-dimensional flows. Obtain the complex potential of a doublet in two dimensions. **[10]**

b) Find the equation of the streamlines due to uniform line sources of strength *m* through the points A($-c$, 0), B($c$, 0) and a uniform line sink of strength 2m through the origin. **[6]**

*Q6)* a) State and prove the theorem of Kutta and Joukowski. **[8]**

b) When an infinite liquid contains two parallel equal and opposite vortices at a distance 2b, prove that the streamlines relative to the vortices are given by the equation $\log\left\{\dfrac{x^2+(y-b)^2}{x^2+(y+b)^2}\right\}+\dfrac{y}{b}=c$ the origin being the middle point of the join, which is taken for axis of *y*. **[8]**

*Q7)* a) Write a note on Stoke's stream function and explain its physical meaning. **[8]**

b) Find the stream function of two-dimensional motion due to two equal sources and an equal sink midway between them, sketch the streamlines and find the velocity at any point. **[8]**

*Q8)* a) Define viscosity and show that the general motion of a fluid particle consists of a pure strain and a rotation. **[10]**

b) What is the complex potential for two-dimensional fluid motion? Discuss the flow for which $w = z^2$. **[6]**

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**Total No. of Questions : 8]** **[Total No. of Pages : 3**

# P823

## [3621] - 403

## M.A. / M.Sc.

## MATHEMATICS

## MT - 803 : Measure and Integration

***Time : 3 Hours]*** ***[Max. Marks : 80***

***Instructions to the candidates:***

1) ***Attempt any five questions.***
2) ***All questions carry equal marks.***
2) ***Figures to the right indicate full marks.***
4) ***(X, B, μ) is a measure space.***

***Q1)*** a) Suppose that for each $\alpha \in Q$, there is assigned a set $B_\alpha \in B$ such that $B_\alpha \subset B_\beta$ if $\alpha < \beta$. Prove that there exists a unique measurable extended real valued function $f$ on X such that $f \leq \alpha$ on $B_\alpha$ and $f \geq \alpha$ on $X \setminus B_\alpha$. **[8]**

b) Let $f_n, f : (X, B, \mu) \rightarrow R$.

i) When does $f_n$ converge to $f$ in measure?

ii) If $f_n$ converges to $f$ in measure, then there is a subsequence $f_{nk}$ that converges to $f$ almost every where. **[8]**

***Q2)*** a) Prove the lebesgue dominated convergence theorem. **[4]**

b) If $f$ is integrable with respect to μ, then given $\varepsilon > 0$, there is a simple function Q such that $\int_x |f - Q| d\mu < \varepsilon$. **[6]**

c) Define $f$ ; $R \rightarrow R$ by $f(x) = x$ if $0 \leq x \leq 3$

$= 0$ otherwise.

Find a simple function Q such that $\int_R |f - Q| dx < \frac{1}{2}$. **[6]**

***Q3)*** a) Give an example of a signed measure of R for which [1,2] is a positive set and [3, 4] is a negative set. **[4]**

b) State and prove the Hahn decomposition theorem. **[8]**

c) If $v$ is a signed measure on (X, B), define the measure $|v|$. **[4]**

***P.T.O.***

*Q4)* a) Prove the lebesgue decomposition theorem for σ-finite measures. **[8]**

b) If μ and $v$ are σ-finite and $v << \mu$, and $f$ is a non negative measurable function, prove that $\int\limits_x f dv = \int\limits_x f \cdot \left(\frac{dv}{d\mu}\right) \cdot d\mu$. **[4]**

c) If $v$ is a signed measure such that $v \perp \mu$ and $v << \mu$, prove that $v = 0$. **[4]**

*Q5)* a) Let (X, B, μ) be a finite measure space, and $g$ an integrable function such that for some constant M, $|\int_x gQd\mu| \le M\|Q\|_p$ for all simple functions Q. Prove that $g \in L^q$ (X, B, μ), where $1 \le p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$. **[8]**

b) Give an example of a continuous linear map on

i) $L^\infty(X, B, \mu)$

ii) $L^1(X, B, \mu)$

iii) $L^2(X, B, \mu)$ **[8]**

*Q6)* a) What is a measure on an algebra of sets? Give an example. **[4]**

b) Let μ be a measure on an algebra $\mathfrak{A}$ on X, μ* the outer measure induced by μ, and $E \subseteq X$. Prove that for $\varepsilon > 0$, there is a set $A \in \mathfrak{A}_\sigma$, with ECA and $\mu^*(A) \le \mu^*(E) + \varepsilon$. **[8]**

[Here $\mathfrak{A}_\sigma$ = set of countable unions of sets of $\mathfrak{A}$ ].

c) Define the product measure on a product of measure spaces. **[4]**

*Q7)* a) State TonelliŚ theorem. **[4]**

b) Let $f$ and $g$ be integrable functions on X and Y respectively, and define $h(x,y) = f(x)\ g(y)$. Prove that $f$ is integrable on X × Y and

$$\int\limits_{X\times y} hd(\mu\times v) = (\int\limits_X fd\mu)\cdot(\int\limits_Y gdv)$$ **[6]**

c) If $f, g \in L^{'}(R)$, let $(f * g)(x) = \int\limits_R f(x-t)g(t)dt$ and $\hat{f}(s) = \int\limits_R e^{ist} f(t)dt$.

i) Prove that $\hat{f}$ is a bounded complex function.

ii) Prove that $(f * g)^\wedge = \hat{f}.\hat{g}$. **[6]**

***Q8)*** a) Define the Hausdorff $\alpha$-dimensional measure on a metric space. **[4]**

b) If X is a locally compact Hausdorff space, how do you construct the Baire sets and the Borel sets? **[4]**

c) State the Riesz representation theorem. **[4]**

d) Prove the Riesz representation theorem. **[4]**

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**Total No. of Questions : 8]** **[Total No. of Pages : 2**

**P824**

# [3621] - 404
# M.A. / M.Sc.
# MATHEMATICS
# MT - 804 : Mathematical Method - II

***Time : 3 Hours]*** ***[Max. Marks : 80***

***Instructions to the candidates:-***

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

***Q1)*** a) Under the suitable conditions prove that

$$\int_a^{s_n}\int_a^{s_{n-1}}.....\int_a^{s_3}\int_a^{s_2} f(s_1)ds_1 ds_2...ds_n = \frac{1}{(n-1)!}\int_a^s (s-t)^{n-1} f(t)dt$$ **[8]**

b) Convert the initial value problem $y''(s) - \sin sy'(s) + e^s y(s) = s$ with the initial condition $y(0) = y'(0) = -1$ to an integral equation of the form $y(s) = f(s) + \int_0^s k(s,t)\,y(t)dt$. Verify your answer. **[8]**

***Q2)*** a) Find the Green's function for the boundary value problem $y'' + \mu^2 y = 0$; $y(0) = y(1) = 0$, $\mu \neq 0$. **[8]**

b) Solve the Fredholm integral equation of second kind

$g(s) = s + \lambda \int_0^1 (st^2 + s^2 t)\; g(t)dt.$ **[8]**

***Q3)*** a) Show that the eigen function of symmetric kernel, corresponding to different eigen values are orthogonal. **[8]**

b) Find the resolvent kernel of the voltera integral equation with the kernel $k(x, t) = 1$. **[8]**

***Q4)*** a) Solve $\phi(x) = 25 + 6x + \int_0^x [5-6(x-t)]\phi(t)dt.$ **[8]**

b) Find the Neumann series for the solution of the integral equation $g(x) = 1 + x + \lambda \int_0^x (x-t)\, g(t)dt.$ **[8]**

*P.T.O.*

*Q5)* a) Define : **[6]**

i) Functional

ii) Basic lemma of calculus of variation.

iii) Hamilton's principle.

b) State and prove the Euler-Lagrange's Equation. **[10]**

*Q6)* a) Determine the curve $y = y(x)$ for which the functional $I = \int_0^{\pi/2} (y'^2 - y^2)dx$; $y(0) = 0$; $y(\pi/2) = 1$ is in extremal. **[8]**

b) Use the variational method to show that the shortest distance between two points in an Euclidean space $E_3$ is a straight line. **[8]**

*Q7)* a) Find the extremizing function of an Isoperimetric problem for getting the extremal of the functional

$I = \int_0^1 (y'^2 + z'^2 - 4xz' - 4z)dx$ under the conditions

$\int_0^1 (y'^2 - xy' - z'^2)dx = 2$ and $y(0) = 0 = z(0)$; $y(1) = 1 = z(1)$. **[10]**

b) Use Hamilton's principle to find the equations of motion of a single particle of mass $m$ moving under gravity. **[6]**

*Q8)* a) The faces $x = 0$ and $x = c$ of a rectangular bar are kept at zero temperature and the initially temperature at any point inside the bar is function of $x$ then find the temperature formula i.e. find $u(x,t)$ if $u_t(x,t) = ku_{xx}(x,t)$ $(0 < x < c, t > 0)$ subject to condition $u(0,t) = 0$; $u(c,t) = 0$, $u(x, 0) = f(x)$, $0 < x < c$. **[10]**

b) Solve $\int_0^x e^{x-t} \phi(t)dt = \sin x$. **[6]**

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**Total No. of Questions : 8]** **[Total No. of Pages : 2**

# P825

## [3621] - 405
## M.A. / M.Sc.
## MATHEMATICS
## MT - 805 : Field Theory

***Time : 3 Hours]*** ***[Max. Marks : 80***

***Instructions to the candidates:-***

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

***Q1)*** a) Let E be a field and F be a subfield of E. If E is a finite extension of F then prove that E is algebraic over F. **[4]**

b) Give an example of an algebraic extension which is not a finite extension. Justify! **[4]**

c) Let E = Q($\alpha$), where $\alpha$ is a root of the equation $x^3 + x^2 + x + 2 = 0$. Express $(\alpha^2 + \alpha + 1)(\alpha^2 + \alpha)$ and $(\alpha - 1)^{-1}$ in the form $a\alpha^2 + b\alpha + c$ with $a, b, c \in$ Q. **[4]**

d) Let F be a field and E = F($\alpha$). If $\alpha$ is algebraic over F of odd degree, show that E = F($\alpha^2$) **[4]**

***Q2)*** a) Let K be a field and $f$ be a non-constant polynomial in K[$x$]. Prove that there exist an extension E of K which contains a root of $f$. **[6]**

b) Find the splitting field of the polynomial $x^4 - 2$ over Q. Also, find the degree of extension. **[5]**

c) If $\alpha$ is a complex root of $x^6 + x^3 + 1$ then find all the homomorphisms $\sigma : Q(\alpha) \to C$. **[5]**

***Q3)*** a) Let F be a field and K/F be a finite separable extension. Prove that there is $\alpha \in$ K such that K = F($\alpha$). **[7]**

b) Let $\alpha$ be a real number such that $\alpha^4 = 5$. **[5]**

i) Show that Q($i\alpha^2$) is normal over Q.

ii) Show that Q($\alpha + i\alpha$) is not normal over Q.

c) Show that $\sqrt{2}+\sqrt{3}$ is algebraic over Q. Find its degree over Q. **[4]**

*Q4)* a) Let $p$ be a prime and $F_p = Z/pZ$. Suppose $F_p^a$ denote an algebraic closure of $F_p$. Prove that for each integer $n \geq 1$ there exists a finite field of order $p^n$ denoted by $F_{p^n}$ uniquely determined as a subfield of $F_p^a$. **[7]**

b) Give an example of an inseparable extension. Justify your answer. **[4]**

c) Let $p$ be a prime and $q = p^\alpha$. Let $F_q$ be the finite field with $q$ elements. Determine the group of all automorphisms of $F_q$. **[5]**

*Q5)* a) Let K be a Galois extension of k with group G. Let F be a subfield, $k \subset F \subset K$, and $H = G(K/F)$. Prove that F is normal over k if and only if H is normal in G. **[8]**

b) Show that the Galois group of $X^3 - X + 1 \in Q[x]$ is $S_3$ while the Galois group of $X^3 - 3X + 1 \in Q[x]$ is cyclic of order 3. **[6]**

c) Is the following statement true or false? Justify! C is algebraic closure of Q. **[2]**

*Q6)* a) Let y be a primitive $n^{th}$ root of unity. Show that $Q(y)/Q$ is a Galois extension.

Find Galois group of $Q(y)$ over Q. **[10]**

b) Let E be a finite extension of $k$ and $\alpha \in E$. Define norm and trace of $\alpha$. Further, if $k = Z/7Z$ and E be the splitting field of the polynomial $X^2 + 1$ then find norm and trace of $\alpha$,where $\alpha^2 + 1 = 0$, $(\alpha \in E)$ **[6]**

*Q7)* a) Let $k$ be a field of characteristic $p$ and K be a cyclic extension of $k$ of degree $p$. Prove that there exists $\alpha \in K$ such that $K = k(\alpha)$ and $\alpha$ satisfies an equation $X^p - X - a = 0$ with some $a \in k$. **[7]**

b) Let $\Phi_n(X)$ denote the n-th cyclotomic polynomial.

Prove that $\Phi_{p^r}(X) = \Phi_{p^{r-1}}(X^{p^{r-1}})$, where $r \geq 1$ is an integer. **[5]**

c) Let $k$ be a finite extension of the rationals. Show that there is only a finite number of roots of unity in k. **[4]**

*Q8)* a) Let F be a field of characteristic p, $p \neq 2$, $p \neq 3$. If $f(x) \in F[x]$ is an irreducible polynomial of degree less than or equal to 4. Let K be the splitting field of $f(x)$. Prove that K/F is a Galois extension. Is K solvable by radicals? **[8]**

b) Let E be a finite separable extension of k. Prove that $T_r : E \to k$ is a non-zero functional. Further, show that the map $(x, y) \mapsto T_r(xy)$ of $E \times E \to k$ is bilinear and identifies E with its dual space. **[8]**

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**P826**

# [3621] - 406
# M.A / M.Sc.
# MATHEMATICS
## MT 806 : Lattice Theory

*Time : 3 Hours]* *[Max. Marks :80*

***Instructions to the candidates:***

1) *Attempt any five questions.*
2) *Figures to the right indicate marks.*

***Q1)*** a) Prove that the set $A$ of all real valued functions defined on $X$ is a lattice with respect to the relation: for $f, g \in A$, set $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$. **[4]**

b) Let $< P; \leq >$ be a poset in which $inf\ H$ exists for all $H \subseteq P$. Show that $< P; \leq >$ is a lattice. **[4]**

c) Let the algebra $\mathcal{L} = < \mathcal{L}; \wedge, \vee >$ be a lattice. Set $a \leq b$ if and only if $a \wedge b = a$. Then prove that $\mathcal{L}^P = < \mathcal{L}; \leq >$ is a poset, and the poset $\mathcal{L}^P$ is a lattice. **[8]**

***Q2)*** a) Define a congruence relation and show that a reflexive binary relation $\theta$ on a lattice $L$ is a congruence relation if and only if the following three properties are satisfied, for $x, y, z \in L$:

i) $x \equiv y\ (\theta)$ if and only if $x \wedge y = x \vee y\ (\theta)$.

ii) $x \leq y \leq z$, $x \equiv y\ (\theta)$, and $y \equiv z\ (\theta)$ imply that $x \equiv z\ (\theta)$.

iii) $x \leq y$, $x \equiv y\ (\theta)$ imply that $x \wedge t \equiv y \wedge t\ (\theta)$ and $x \vee t \equiv y \vee t\ (\theta)$. **[10]**

b) Prove that $I$ is a prime ideal of a lattice $L$ if and only if there is a homomorphism $\phi$ of $L$ onto $C_2$ with $I = \phi^{-1}\ \{0\}$. **[6]**

***Q3)*** a) Show that if the chain with six elements $C_6 \cong L \times K$ then $L$ or $K$ has only one element. **[5]**

b) Prove that L is distributive if and only if the identity
$(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$.
holds in $L$. **[5]**

c) Let I be an ideal and let D be a dual ideal. If $I \cap D \neq \phi$ then show that $I \cap D$ is a convex sublattice, and every convex sublattice can be expressed in this form in one and only one way. **[6]**

***Q4)*** a) Let $L$ be a distributive lattice with 0. Show that $Id(L)$ is pseudocomplemented. Is the converse true? Justify. **[8]**

b) State and prove Stone's Theorem for separation of distributive lattices. **[8]**

***Q5)*** a) Prove that lattice is modular if and only if it does not contain a pentagon as a sublattice. **[8]**

b) Prove that a lattice is distributive if and only if every element has at most one relative complement in any interval containing it. **[4]**

c) Prove that every ideal of a distributive lattice is a neutral ideal. **[4]**

***Q6)*** a) Let $L$ be a bounded distributive lattice with $0 \neq 1$. Then prove that $L$ is Boolean if and only if the set of prime ideals $P(L)$ is unordered. **[8]**

b) State and prove Hashimoto Theorem. **[8]**

***Q7)*** a) Prove that in a modular lattice an element is standard if and only if it is distributive. **[8]**

b) Prove that in a Boolean lattice an ideal is maximal if and only if it is prime. **[8]**

***Q8)*** a) Prove that if $L$ is finite then $L$ and Id($L$) are isomorphic. **[6]**

b) Prove that the set of all neutral elements of a lattice forms a sublattice. **[6]**

c) Find all central elements of $C_2 \times C_3$. **[4]**

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Total No. of Questions : 8] [Total No. of Pages :3

**P827**

# [3621] - 407
# M.A / M.Sc.
# MATHEMATICS
## MT - 807 : Combinatorics

*Time : 3 Hours]* *[Max. Marks :80*

***Instructions to the candidates:***

1) *Answer any five questions.*
2) *Figures to the right indicate marks.*

***Q1)*** a) What is the probability that the top two cards in a shuffled deck do not form a pair? **[4]**

b) How many 8-digit sequences are there involving exactly six different digits? **[4]**

c) How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 12$ with $x_i \geq 0$? How many solutions with $x_i \geq 1$? **[4]**

d) How many ways are to distribute 20 different toys among five children without any restriction. **[4]**

***Q2)*** a) Verify the following identity by a combinatorial argument

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$ **[6]**

b) Verify the following identity by block walking

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \text{..........} + \binom{n}{r} = \binom{n+1}{r+1}.$$ **[6]**

c) If $\binom{n}{3} + \binom{n+2}{3} = {}^{n}P_{3}$, find $n$. **[4]**

***Q3)*** a) i) Build a generating function for $a_r$, the number of $r$ selections from a pile of seven types of lightbulbs with an odd number of the first and second type.

ii) Build a generating function for $a_r$, the number of integer solutions to the equation

$$e_1 + e_2 + e_3 = r, \qquad 0 < e_i < 4.$$ **[6]**

***P.T.O.***

b) Use generating functions to find the number of ways to distribute 25 identical balls into seven distinct boxes if the first box can have no more than 10 balls but any number can go into each of the other six boxes? **[6]**

c) Find a generating function for $a_r$, the number of partitions of $r$ into even integers. **[4]**

***Q4)*** a) Use exponential generating functions to find the number of r-digit quaternary sequences with an even number of 0s and an odd number of 1s. **[8]**

b) Find an ordinary generating function whose r[th] coefficient $a_r$ is $3r + 7$. Hence evaluate the sum

$$7 + 10 + 13 + \ldots\ldots + (3n + 7).$$ **[8]**

***Q5)*** a) Find and solve the recurrence relation for the number of regions into which the plane is divided when $n$ straight lines are drawn in such a way that every pair of lines intersect (but no three lines intersect at a common point). **[8]**

b) Solve the recurrence relations :

i) $a_n = 2\,a_{n-1} + 1,\ \ a_1 = 1.$

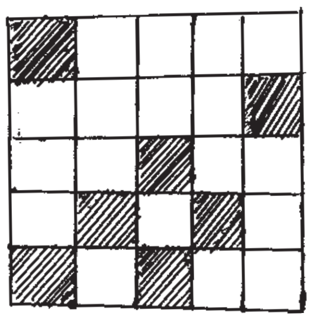
ii) $a_n = a_{n-1} + 3n^2,\ \ a_0 = 10.$ **[8]**

***Q6)*** a) State and prove the Inclusion - Exclusion Principal. **[8]**

b) Use the principal in part (a), to find the number of different integer solutions to the equation

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 20, \qquad 0 \le x_i \le 8.$ **[8]**

***Q7)*** a) Find the rook polynomial for the following board.

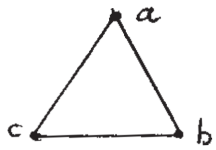


**[8]**

b) Use rook polynomials to find the number of ways to send six different birthday cards, denoted $C_1$, $C_2$, $C_3$, $C_4$, $C_5$, $C_6$ to three aunts and three uncles, denoted $A_1$, $A_2$, $A_3$, $U_1$, $U_2$, $U_3$ if aunt $A_1$ would not like cards $C_2$ and $C_4$; if $A_2$ would not like $C_1$ or $C_5$; if $A_3$ likes all cards; if $U_1$ would not like $C_1$ or $C_5$; if $U_2$ would not like $C_4$, and if $U_3$ would not like $C_5$. **[8]**

***Q8)*** a) i) Prove that for any two permutations $\pi_i$, $\pi_j$ in a group G there exists a unique permutation $\pi_k = \pi_i^{-1} . \pi_j$ in G such that $\pi_i . \pi_j = \pi_k$.

ii) Find all symmetries of the following figure.




**[8]**

b) State Polya's enumeration formula. Use this formula to determine the pattern inventory for 3-bead necklaces distinct under rotation using black and white beads. **[8]**

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