BROUWER'S UNDERSTANDING OF THE LOGICAL CONSTANTS

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1. Verificationist Interpretations : Meaning as Provability

Most attempts to define the intended interpretation of the intuitionistic logical operators take as the basic key concept the notion of 'proof'.

The idea is to equate the meaning of the mathematical statement with its provability coniditions. That is: to give the interpretation of a statement by means of definition of what is to be a proof of it.

This seems more adequate for intuitionistic semantics given that, in contrast with the structural (platonistic) point of view, intuitionistic mathematics focuses primarily on the *subject* (the creative mathematician) and his ability to perform certain mathematical operations by applying his previously designed constructions (*knowing how*). Hence a notion such as 'proof', which refers to the successful completion of a human action, appears to be more suitable than that of 'truth'.

On the other hand, classical mathematics focuses essentially on the *object*: eternal pre-existing mathematical structures (*knowing that*); and for this reason, the notion of 'truth', with its prominent descriptive untensed character, is more appropriate. Of course we can also use the predicate 'true' with its intuitionistic sense, e.g. as 'having been proved', as many intuitionistic mathematicians do, but that could be misleading to a classical mathematician if we do not make our intention explicit, especially in the context of a semantic definition.

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The idea that intuitionistic meaning should be equated with the proof-conditions is implicit in many of Brouwer's writings, although he never states it directly. The following quotation is a relatively clear illustration. It is taken from the *Cambridge Lectures*, but the same idea is expressed in many others of his papers, sometimes with almost the same words (cf.e.g. (1955), pp. 551-52) :

> "Classical logic presupposed that indepenetdently of human thought there is a turth, part of which is expressible by means of sentences called 'true assertions', mainly assigning certain properties to certain objects or stating that objects possessing certain proeperties exist or that certain phenomena behave according to certain laws.
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> "(...) Only after mathematics had been recognized as an autonomous interior constructional activity(...) the criterion of truth and falsehood of mathematical assertion was confined to mathematical activity itself, without appeal to logic or to hypothetical omniscient beings. An immediate cosequence was that for a mathematical assertion A the two cases of truth and falsehood, formerly exclusively admitted, were replaced by the following three:
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> (1) A has been proved to be true;
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> (2) A has been proved to be absurd;
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> (3) A has neither been proved to be true nor to be absurd, nor do we know a finite algorithm leading to the statement either that A is true or that A is absurd." (Brouwer (1981), pp. 90-92. I always quote using my choice of notation).
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> In a footnote he adds: "the case that A has neither been proved to be true nor to be absurd, but that we know a finite algorithm leading to the statement either that A is ture, or that A is absurd, obviously is reducible to the first and second cases" (p.92).

2. *Verificationist interpretations : Meaning as Problem-solving*

A variant on this view is to consider that every mathematica! statement is the statement of a problem, to be solved either positively or negatively. This is the idea exploited in Kolmogorov's interpretation [Kolmogorov, (1932)], and is also suggested more or less directly by Brouwer's writings in some places.

For example in (1928) Brouwer urges the formalists to accept "(...) the identification of the principle of excluded middle with the principle of the solvability of every mathematical problem." (p.491).

This identification means that the principle of excluded middle holds if and only if each mathematical problem is solvable. In the direction from right to left the implication is obvious: if every mathematical problem were solvable then the principle of excluded middle could not fail to hold. However, the implication in the other direction, that is, from excluded middle to the principle of solvability, seems to entail that the meaning of each statement is the formulation of a mathematical problem. Otherwise we clould not understand why the fact that either a statement or its negation holds entails that the respective problem is solvable.

Of course in classical mathematics we can also assign to each statement a corresponding problem: the problem of proving that what the statement affirms is true; but that would only be an oblique interpretation with respect to the primary meaning of the statment, which, e.g. under the platonistic view, would be the statment of a mathematical 'fact'.

3. *The Operational Interpretation*

Finally, in Brouwer's writings we also find support for a more basic explanation of the meaning of mathematical statements, in terms of elementary manipulations with mathematical constructions, I shall call this interpretation 'operational'. The term was suggested by Professor Machover; it appeared in Prawitz, (1973, p.231) in reference to the verificationist interpretation, but it has not been used again since then.

The idea of the operational interpretation is that a mathematical statement expresses an expectation that the result of performing a particular construction will satisfy certain properties, or better, that it will *agree* with the constructions corresponding to those properties, if they are also completely effected. This idea is very well-known, and indeed essential to intuitionism and to Brouwer's thinking, Here are a couple of quot, from Brouwer's PhD Dissertation and from (1923) :

> "Often it is quite simple to construct inside such a structure, independently of how it originated, new structures, as the elements of which we take elements of the original structure or systems of these, arranged in a new way, but bearing in mind their original arrangement. The so-called 'properties' of a system express the possibility of constructing such new systems having a certain connection with the given system.
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> "And it is exactly this *imbedding* of new systems in a given system that plays an important part in building up mathematics (...)". [1907, p.52].
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> "Within the limits of a definite finite main system one can always test, that is prove or reduce to absurdity the properties of the system, i.e. test whether a system can be fitted into another according to prescribed incidence of elements since the fitting-in as determined by the property can in every case be executed in only a finite number of ways, which each in turn can be undertaken and pursued either until it is successfully completed or until it gets stuck." [1923, p.235 ; this translation, however, is from van Stigt (1990), P.243].

In the case of a numerical equality over natural numebr, for example, this would mean that the two completed constructions fit perfectly well into each other, for example '$12^2 = 144$' would mean that the result of effecting both sides of the equality comes to the same final construction.

In the case of an atomic statement other than an equality, this would mean that the constructions corresponding to the objects involved satisfy the construction corresponding to the relation in question; for example the meaning of '$2^{11213} - 1$ is prime' would be that the result of effecting the operation $2^{11213} - 1$ would satisfy positively the constructive procedure to test the property of 'being prime'.

On the other hand, in the case of more complex statements, the expectation would be that some simpler constructions can be connected according to the main logical operator, which itself would be a constructive procedure of some kind.

4. *The Verificationist versus the Operational Interpretation*

The operational explanation is fairly close to the provability or verificationist interpretation, and has been assimilated to it by most authors. The point would be that the proof of a statement consists precisely in producing the mathematical construction which the statement demands.

Brouwer himself seems to support this identification when he writes, for example, "the words of your mathematical demonstration merely accompany a mathematical construction that is effected without words" (1907, p. 73).

However, this identification is correct only in the cases where it is obvious that the construction which has been produced has the required properties in particular, for instance, with proofs of atomic statements–. In general however, the proof of the statement in question will have to include not only the required construction, but also an argument that it is in effect such a construction; and in some cases this argument may necessarily be very complicated.

This is not the time, however, to carry on this discussion. It is enough to notice that the two interpretations may not coincide, and that we should not equate them beforehand.

5. *The Meaning of Negation*

As Brouwer made clear many times, for him to negate a mathematical statement was to claim the absurdity or impossibility of what the statement says. Apparently, Brouwer was not the first person to conceive negation in this way; it seems that similar definitions had been given before at least by Husserl and Oskar Becket [cf. Heyting (1931) p.59].

Indeed, Brouwer often writes ' is absurd' and similar phrases instead of 'not' or is false'; for example, he formulates the law of excluded middle as the principle that every property is either correct or impossible (1923, p. 335).

In (1949) he says: "by non-equivalence we understand absurdity of equivalence, just as by noncontradictority we understand absurdity of contradictority (p.95, footnote 2); and sometimes he even writes "false i.e. absurd" (e.g. (1955), p. 552).

In classical mathematics to say that a statement is 'absurd' means that it is obviously false, something which has to do more with the psychological perception of the statment than with the statement itself. 'Absurdity' in this sense, is not a proper technical term of classical mathematics.

In intuitionistic mathematics, on the contrary, 'absurdity' is the most interesting way of expressing negation. The absurdity of a hypothetical consturction means that not only is it difficult to effect it – because it required great ingenuity or hard work – but that it is intrinsically impossible, so that we shall no longer bother to attmept it.

Sometimes the absurdity is plain to see and does not need any demonstration (e.g. '1 =2': it is obvious that the two constructions do not 'agree' or fit into each other). Some other times the absurdity is not obvious but we can find a way of reducing the construcions in question to a point where the impossibility becomes clear.

It is for this reason that some later authors define a negation $\neg A$ as the conditional statement $A \rightarrow \perp$, where $\perp$ is a fixed absurdity, In fact $\perp$ is often called a 'contradiction', but that must not be interpreted in the classical sense (e.g. as any statement of the form $(B \wedge \neg B)$, because then it would be obviously vacuous.

Instead, we can take $\perp$ as a basic absurdity, such as "1 = 2', whose only role is to make absolutely evident that the construction which has been reduced to it is impossible: "at the point where you enounce the contradiction, I simply perceive that the construction no longer *goes*, that the required structure cannot be imbedded in the given basic structure." (1907, p.73).

Later, Dummett (1977) has suggested that given a decidable atomic statement B we could identify $\bot$ with $B \wedge \neg B$, where the meaning of $\neg B$ would be given directly by the decision procedure, attached to it. Then, anyone who understands the decision procedure will recognize that it is impossible for it to give two opposing results, and hence that whichever construction that has been reduced to such a statement is also impossible.

Other authors; in a vaguely similar way to Kripke semantics, have defined a proof of $\neg A$ directly as 'a proof that there cannot be a proof of A' (Bell and Machover (1977), p. 406, and Dummett himself in (1976), p.110). However, it is difficult to make constructive sense of this idea independently of the reduction to basic absurdity. For, in general, the impossibility of a complicated construction will not be plain to see, and will have to be shown by means of reduction of this construction to another, elementary one, whose absurdity is obvious.

On the other hand, the claim is not simply that *as a matter of fact* we shall never be able to perform A–e.g. because A is too complicated –, but that A is intrinsically impossible. However, we cannot admit a priori this type of impossibility because that would imply a certain reification over the universe of constructions; constructions are not assumed to exit or not (to be possible or impossible) independently of us – independently of our ability to prove it so–.

Hence the only way of making sense of this idea is, again, by means of a reduction of A to a basic impossibility.

6. *Negation and Hypothetical Constructions*

As for hypothetical constructions, there is a passage in Brouwer's PhD Dissertation which has misled some into believing that he rejected them:

> "In one particular case the chain of syllogisms is of a somewhat different kind, which seems to come nearer to the usual logical figures and which actually seems to presuppose the hypothetical judgement from logic. This occurs when a structure is defined by some relation in another structure, while it is not immediately clear

> how to effect its construction. Here it seems that the construction is supposed to be effected, and that starting from this hypothesis a chain of hypothetical judgements is deduced. But this is no more than apparent; what actually happens is the following: one starts by setting up a structure which fulfils part of the required relations, thereupon one tries to deduce from these relations, by means of tautologies, other relations, in such a way that these new relations, combined with those that have not yet been used, yield a system of conditions, suitable as a starting-point for the construction of the required structure. Only by this construction will it be proved that the original conditions can be fulfilled." (1907, p.72)

However, he is not condemning the appeal to hypothetical constructions in general, but only the assumption that a mathematical construction can exist without us having first *proved* that it could be effected and how. Moreover, he himself often referred to hypothetical constructions in his proof of negation and conditional statements (e.g. in his proof of the law of triple negation [1981, p.12].

Later Freudenthal (1937) and Griss (1946) criticized the use of hypothetical constructions especially in the cases where the supposed construction turns out to be impossible, as happens in a proof of a negation, if the proof is successful. This led Griss to the extreme position of trying to develop intuitionstic mathematics without using negation at all (1946), (1955).

Heyting (1937), (1961), on the contrary, defended the use of hypothetical constructions in mathematical reasoning:

> "The following simple example shows that the problem $A \rightarrow B$ in certain cases can be solved without a solution for the problem A being known. For A I take the problem 'find in the sequence of decimals of π a sequence 0123456789', for B the problem 'find in the sequence of decimals of π a sequence 012345678'. Clearly B can be reduced to A by a very simple construction." (1937, p. 117; the translation is from Troelstra and van Dalen (1988, p.31)].

Thus, by reducing the negation to an absurdity operator, and more in general, to the construction which shows that absurdity, the intuitionistic mathematician manages to assign a *positive* meaning to each negation statement, in accordance with the constructive philosophy of mathematics. An intuitionistic negation is strictly speaking a positive claim – that which reduces the hypothetical construction to a basic impossibility such as '1 = 2' –, but it carries with it an implicit denial – the denial that we shall ever be able to perform the construction corresponding to the statement negated –. Moreover, this will be obvious to anyone who understands the absurdity of the basic impossibility in question - e.g. the absurdity of ' 1= 2'.

7. *The Meaning of the Conditional*

Brouwer's conception of the conditional in its strongest sense appears in his attempted proof of the bar therorem (e.g. in (1927); a neat exposition and discussion is Dummett (1977, pp. 94-104). There, Brouwer considers a conditional statement (bar induction), classifies all possible proofs of the antecedent into three types, and tries to show that each of these proofs can be converted into a proof of the consequent.

This suggests that an intuitionistic proof of a conditional statement A → B is a method of transforming every proof of *A* into a proof of *B*.

An obvious question, however, is how can we know in advance which form any arbitrary proof of the antecedent should take, so that we ensure that our method will transform all of them into proofs of the consequent. This question turns out to be a deep one.

We must notice that, in particular, as it happens, Brouwer's attempted proof of the bar theorem is incorrect, and no way has been found to correct it while preserving its original form (see again Dummett (1977) pp. 101-102).

In practice, most intuitionistic proofs of conditional statments appeal to oņly one obvious proeprty that every proof of the antecedent must satisfy: *to be a proof of the antecedent* - that is, to have the antecedent as the final line or conclusion of the proof –. The method then does not enter to transform the proofs of *A* internally, but simply *extends* them to obtain proofs of *B*.

Brouwer himself, in other proofs of conditional statements returns to this simple procedure. For example, in his proof of the law of *triple* negation (e.g. (1981), p.12). he first assumes $\neg\neg\neg A$ (that is, $\neg\neg A \rightarrow \bot$) and then show how we can transform that construction into a proof of $\neg A$ (that is, into a proof of $A \rightarrow \bot$), independently of any actual proof of the former. Moreover, every notable intuitionstic proof of a conditional statement has proceeded in a similar way as well (Dummett (1977), pp. 15 and 104).

8. *Disjunction, Conjunction, and* $\leftrightarrow$

From his discussion of the law of excluded middle we can see that in order to accept a disjunction Brouwer required that one of the disjuncts in known to hold — or at least that a decision procedure is known which could be used to determine which one —.This is something on which he insists many times. The following quote provides an illustration:

> "Now consider the *principium tertii exclusi*: it claims that every supposition is either true or false; in mathematics this means that for every supposed inbedding of a system into another, satisfying certain given conditions, we can either accomplish such an imbedding by a construction, or we can arrive by a construction at the arrestment of the process which would lead to the imbedding. It follows that the question of the validity of the *principium tertii exclusi* is equivalent to the question *whether unsolvable mathematical problems can exist*. There is not a shred of a proof for the conviction, which has sometimes been put forward that there exist no unsolvable mathematical problems.
>
> "Insofar as only finite discrete systems are introduced, the investigation whether an imbedding is possible or not, can always be carried out and admits a definite result, so in this case the *principium tertii exclusi* is reliable as a principle of reasoning." (1908, pp. 109).

However "in infinite systems the *principium tertii exclusi* is as yet not reliable " (p.110).

In this way Brouwer succeeds in attaching a constructive meaning to disjunction statements: to assert a disjunction, the subject must be able to perform the constructions corresponding to one of the disjuncts.

Conjunction shall not detain us long.This is indeed, among the five logical operators, the only one which essentially does not change its meaning. except for the fact that it is now embedded in an intuitionistic language, and the other logical operators to which it relates are different from those of classical mathematics.

For that matter, the biconditional is also *defined as in classical logic* - e.g. '$A \leftrightarrow B$' is an abbreviation of ' $(A \rightarrow B)$ $(B \rightarrow A)$'. This is not to say that intuitionistically '$A \leftrightarrow B$" means the same as in classical logic, because again both '$A \rightarrow B$' nad '$B \rightarrow A$' have changed their meaning with respect to classical logic.

Similarly, 'A $\wedge$ B' does not mean the same, because A and B will have also changed their meaning with respect to their classical counterparts.

9. *The Quantifiers*

Brouwer's conception of the existential quantifier is probably the most characteristic of all the logical operators. Intuitionistically mathematical objects are not assumed to exist by themselves, but only as a result of a generation of construction process. To prove that a certain entity satisfying a given condition exists, it is not enough, for example, to reduce the hypothesis that it did not exist to a contradiction: we must actually produce one, or at least show how it could be produced: " (...) in intuitionist mathematics a mathematical entity is not necessarily predeterminate" (1955, p. 552). This means that $\exists$ can no longer be read as 'there is' in the classical sense – i.e. there exists independently of us –, but rather, as 'we can construct'.

The following quote is an illustration :

"(...) now let us pass to infinite systems and ask for instance if there exists a natural number n such that in the decimal expansion of π the nth, $(n+1)$th, ..., $(n+8)$th, and $(n+9)$th digits form a sequence 0123456789. This question (...) can be answered neither affirmatively nor negatively. But then, from the intuitionist point of view, because outside human thought there are no mathematical truths, the assertion that in the decimal expansion of π a sequence 0123456789 either does or does not occur is devoid of sense." (1981, p.6).

Finally, as for the universal quantifier, Brouwer's requirement for a proof of a universal statement$\forall x A(x)$ was that method had been produced to establish $A(c)$ for each element c in the domain. In particular it would not be enough, as before, to derive a contradiction from the hypothesis that an object d such that $\neg A(d)$ exists; that derivation would not be enough in general to prove $A(c)$ of evey individual c. Instead, he required an effective method – a construction – for doing exactly this.

NOTES

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