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REMARKS ON WITTGENSTEIN'S PHILLOSOPHY OF MATHEMATICS

I) Introduction : (a) Philosophy of mathematics was Wittgenestein's first and the most enduring love. Wittgenestein's introduction to philosophy in general and to the philosophy of Mathematics in particular came when he read Russell's *Principles of Mathematics* in 1909. From that time till 1944 he repeatedly returned to the problems in the field. Even when he thought of problems in ethics and psychology, mathematics was not far removed from his mind.[1]

Philosophy of Mathematics was dominated during this period by the problems thrown up by the discovery of transfinite numbers.It is, therefore, necessary to describe briefly this development. Dedekind (1831-1916) introduced the nation of a set as a collection of different things which can be considered from a common point of view, can be associated in mind.[2] Cantor(1845-1918) formed the Set N ={1,2,3.....} and raised the question : How many elements are there in the set N? Or, in short, what is the number of N?

In counting bananas from a bunch we count each banana against a natural number in the succession 1,2,3.... and if the bunch gives out at number 5 we conclude that the bunch has 5 bananas. Against what willl we count the elements of N ? Cantor said, 'Count them against the numbers 2, 4, 6 ... in succession'. What do we find ? The set N will give out simultaneously when the succession 2, 4, 6 ... gives out.

Consider the set E = { 2, 4, 6, ...}. We counted N against the elements of E. But *prima facie* N has more elements than E; in fact E is a part of N. Still we find that pairing the elements of N and E as 1⟷2, 2⟷4, 3⟷6 etc. shows that N and E give out simultaneously. Therefore, Cantor concluded that the

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number of both N and E is the same and it is an infinite or transfinite number. Cantor called it 'aleph-not'.

Cantor defended the transfinite on the following grounds. 1) We can postulate a world of mathematical entities (which we call the world -3) because it influences the structure of our mind. 2) In the world -3 we are free to introduce new concepts provided they are free from contradiction and stand in a fixed relation to previous concepts. 3) Because of these two conditions on a new concept, its introduction is not likely to cause any harm.[3]

(b) The discovery of the transfinite number had certain novel features.

Firstly, it did not come as a solution of some problem. Thus, the transfinite number had no instrumental value. It came as a result of original and ingenious merging of the notions of counting, number and set. Thus, it appeared to be an autonomous growth of mathematics.

Secondly, when more and larger transfinite numbers were introduced and their arithmetic was constructed, the concept began to exercise a charm. Wittgenstein writes, "If you can show there are numbers bigger than the infinite, your head whirls. This may be the chief reason this was invented".[4] It, therefore, became necessary to legitimise their existence as bonafide mathematical entities.

In merging the notions of counting, number and sets Cantor had bent and distorted them. Thus, counting had been equated to pairing; sets were formed of numbers which were not objects and the number itself was regarded not as sign in the standard succession 1, 2, 3 ... but as the number of a set. The legitmization of the transfinite numbers, thus, demanded a revolution in attitudes to these basic notions of mathematics. Naturally the exercise of incorporating the traansfinite in mathematics was termed as a 'foundational study'.

II) Logicism : (a) Peano (1858-1932) had organised the whole of arithmetic in a deductive system which began with three undefined terms and five axioms.[5] Russell tried to derive this system from another system based on ideas of Cantor.

Russell begins with equivalence of two non-empty sets A and B. They are said to be equivalent if to each element in A corresponds exactlly one element of B and *vice versa.* A class of sets in called an equivalence class if any two sets in the class are equivalent. Each class of equivalent sets is a number which is called a cardinal number to distinguish it from natural numbers. Of course, every natural number turns out to be a cardinal number. But there are cardinals, for example, the equivalence class which has the set N = {1, 2, 3 ...} as a member, which are not natural numbers. Thus, transfinite numbers are now at par with natural numbers and are legitimately accepted as mathematical entities.

Russell now showed that all the three undefined terms of Peano can be defined in terms of set-operations and the axioms of Peano can be proved as theorems. Thus, whole of Peano's arithmetic, arranged in a deductive system, is derivable from the notion of a set and of set operations. So far the question of defining a set did not arise.

(b) Given a set S and an element x, it must be possible to decide if x is a member of S or not. So long as S was finite it was possible to decide it. With the introduction of infinite sets like N it became necessary to lay down a criterion for membership. This criterion is the presennce of a property in x which other members of S possess. In short the set S is given in terms of a predicate. Russell called this the principle of abstraction.[6] A set S consists of objects a, b, c, ... if the replacement of x in the sentence "x is P" by a, b c ... makes it a true statement. This connection between a set and a predicate reduces sets to predicates and set operations to statement calculus.

Replacement of a set by a predicate had an unintended consequence. It led to Russell's paradox. To overcome the paradox Russell introduced axiomatic set theory. This raised the question: what axioms can one take in constructing a theory ? Russell's original criterion was that axioms must be statements which are logically true i.e., tautologies. But at least three axioms in Russell's theory were not logically true.[7] Then the criterion for acceptance of an axiom was decided to be that the axioms must not lead to a contradiction.[8]

In addition, Russell systematized methods of proof, introduced a very extensive symbolism and brought a minute discipline into mathematics. In the present paper we are not going to consider these aspects. (c) This construction of arithmetic from the sets could be regarded as a mathematical theory and in that case Wittgenstein had no quarrel with it.[9] But the logicists made wider claims and about these claims Wittgenstein had strong reservations.

The first claim was that the logicism was a philosophy of mathematics. Russell felt that any science begins with a set of observations which are taken as axioms. What goes forward from the axioms is the subject matter of that science, while what goes backward from the axioms in search of a new set of more reliable and obvious observations is the philosophy of that science. Hence, logicism was a philosophy of mathematics.

In logicism numbers are reduced to sets and sets to predicates. Thus, every mathematical statement is ultimately a formal logical statement. Thus, the logicists had reduced mathematics to formal logic.[10] Since logic is a more fundamental and reliable science, logicism had provided mathematics with a sounder foundation. That is why logicism is a foundational study.

Thirdly, the logicists had improved the method of proof. A proof of a theorem relates the statement of the theorem with the axioms in an indubitable and logically perfect manner. Hence, the truth of a theorem depends on laws of inference and axioms. Therefore, the only harmful thing to mathematics is a possible contradiction between axioms themselves.

Under this straight-jacket formalism, mathematics had become bereft of content. "Pure mathematics consists entirely of assertions that, if such and such proposition is true of anything, then such and such proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is, of which it is true'.[11]

The above consequences flowed from the attitude of the logicists towards mathematics which was summed up by Russell in the following beautiful words. ''Remote from human passions, remote even from pitifull facts of nature, the generations have gradually created an ordered cosmos, where pure thought can dwell as in its natural home."[12]

III) The Beginnings : (a) For Wittgenstein the logicist programme was misconceived in three respects. Firstly, the account of mathematics which it gave was artificial and a-historical. Secondly, what it was designed to achieve, namely the legitimization of the transfinite, was not a worthy goal because the transfinite was an unnatural and cancerous growth on the body of mathematics.[13] Thirdly, the only control on the growth of mathematics, according to the logicists, is the fear of a contradiction. This Wittgenstein finds ridiculous.

We shall beegin by considering Wittgenstein's account of mathematics which will give us an opportunity to dwell upon the first point of his attack on logicism. The latter two points will be discussed when we consider Wittgenstein's conception of control on mathematics.(b) Mathematics begins with techniques. Consider the technique of addition and the statement 2 + 3 = 5. Adding 2 sheep to 3 sheep, adding 2 cows to 3 cows, adding 2 loaves to 3 loaves etc. are empirical processes and the processes result respectively in 5 sheep, 5 cows and 5 loaves. These empirical processes constitute a form of life from which emerges the agreement that 2 + 3 = 5. This is an agreement not in opinion but in the form of life.[14] From several such agreements results the technique of addition.

Technique emerges from experience but later on it is conceived as independent of experience.[15] Once a technique is detached from experiences its rules become timeless and universal.[16] As a code of rules the technique ceases to refer to external objects and the rules become internal relations.[17] Thus, the rule 2 + 3 = 5 becomes a rule about signs 2, 3, 5, and +. Its referents such as sheep, cows or loaves are left out of consideration. A technique which is born out of experience now becomes a standard to judge an experience.[16]

Once a technique is established it grows or extends for various reasons. Thus, new demands are made on the technique, for example the technique of division was asked to carry out the division

8 ÷ 3 and the demand was met by introducing fractions. A simplification in calculations is introduced such as the notion of 'carry' in the technique of addition to facilitate addition of large numbers.

(c) The logicists, however, ignored the historicity of arithmetic altogether and constructed a new technique based on set-operations. A new technique is needed if either (i) the existing technique is found useless in practice or (ii) the new technique has simple methods of calculation e.g. the method of logarithms. On the first count we find that airthmetic was not found wanting in any respect and, moreover, the results in logicist arithmetic are in no way different from those in ordinary arithmetic.[19] As for the second point, it is patently true that logicist calculus is far more laborious than ordinarry arithmetic. "How this calculus of Russell's can be extended you would not know for your life, unless you had ordinary arithmetic in your bones. Russelll doesn't even prove 10 x 1000 = 10,000."[20]

It may be argued that the results in ordinary arithmetic and the logicist arithmetic agree is a coincidence. But results in logicist arithmetic fllow from indubitable logic.[21] When ordinary arithmetic is giving good results, it is well established by custom and when there is no scope for doubt about its efficacy, what do we gain in learning logicist arithmetic? Indeed, if the results of logicist arithmetic are different from those of arithmetic, it is the former that will be in doubt.[22]

Moreover,Russell's arithmetic is based on a particular technique of logic. If there is another logic giving the same results, which one will you take as foundation? We will go "a step further back in order to find something solid which underlies both"[23] the logics and this search will never end. Further, why should one believe that logic is more sacred than arithmetic. It is as much the result of a consensus as ordinary arithmetic. It too is a consensus generated by a shared form of life.[24]

After elaborating this background Wittgenstein concludes : "Why should one want to connect arithmetic with logic? Suppose we said, 'Disregard the connection between arithmetic and logic entirely. Consider arithmetic as a technique which our children learn–perhaps with an abacus.' Isn't that all right? Why hanker after logic?"[25]

IV) *The Growth* : (a) Mathematics grows by adding new techniques to its repertory. Some of these techniques come from the empirical domain as, for instance, arithmetic was isolated from empirical processes. But here we are interested in the growth of mathematics from within. An elementary example of this type of growth is Matrix Algebra.

When a technique comes from the empirical domain its efficacy in the practical life sustains and sanctions it. But many times a technique developed from within mathematics is so remote from experience that it is difficult to substantiate its worthwhileness. Such is the case, for example, with the technique of counting roots of an equation where the same root is counted many times.[26] It is, perhaps, this type of consideraion that led to the postulation of a mathematical realm. "...certain branches of mathematics have been developed in which the charm consists in the fact that pure mathematics looks as though it were applied mathematics – applied to itself. And so, we have this business of a mathematical realm."[27]

The positing of a mathematical realm brings forth the belief that mathematics is a physics of such a realm and that mathematical truths can be discovered by experimenting with the objects in this reality.[28] But this is misconceived. For, an experiement leads to a synthetic statement about reality; it does not give us a rule and mathematical statements are rules.[29] A synthetic statement can be falsified, whereas a rule cannot.

But the conception of a mathematical reality is flawed basically because we will have to postulate different realities corresponding to different techniques. "Similarly, there will be a reality for which 25 x 25 =625 and another in which 25 x 25 = 624 ...The whole thing crumbles down because you are making the assumption that once you are in the right world you will find out."[30]

(b) When the notion of a mathematical reality is rejected we are left with the logicist conception that mathematics is merely a game of signs played with meticulously formulated rules. The signs have no specified referents.[31] But to Wittgenstein this position is totaly unacceptable on two counts. First, the basic techniques are obviouslly derived from real processes and therefore the signs in the technique have real referents. Secondly, new techniques such as, for instance, counting, do seem to describe reality. In fact, had this not been so, a technique would remain a game and not become mathematics.[32]

Here we reach a crucial stage in Wittgenstein's Philosophy of Mathematics. Gerrard Steve observes[33] that Wittgenstein had two philosophies of mathematics. The earlier one based on the calculus conception and the later one based on the language-game conception. We hold that the later philosophy was not based on the language-games. It was a direct development of the earlier one using the language-games as a control on development of mathematics and the key idea here is that of a concept.

(c) Recall that the rule 2 + 3 = 5 was abstracted from several ___________(A)

real world instances like 2 sheep + 3 sheeps = 5 sheeps, 2 sticks + 3 sticks = 5 sticks ___________(I)
___________(II)

etc. This abstraction was the result of an agreement. On what consideration was this agreement based? First, it was observed that there was some family resemblance in the instances I and II. This likeness might not be objective. It was the community which reached the conclusion (A) that might have invented likeness in I and II. Secondly, the like practices (I) and (II) must have operated on like objects. The objects did not multiply when the process was being carried out, for example. Had a sheep calved, the result (I) would not hold. This was not possible with sticks. But sticks and sheep had this likeness viz. that the units survived during the operation. This likeness between a stick and a sheep is not objectively there. It is an invention by the community. Finally, in making the rule (A) from the instances (I) and (II), it was agreed that while there did exist instances where the rule (A) did not work, they are to be ignored in two senses. One, the adverse instance is not to be regarded as a falsifier of the rule

(A) though it could be a falsifier of the synthetic statement (I) and, two, the domain of the rule (A) is so minutely demarcated that adverse instances are automatically barred.[34]

The rule (A), thus, becomes a concept of which instances (I) and (II) are members. Signs 2 and 3 and 5 refer to concepts of which, respectively 2 sheeps, 3 sticks and 5 sheep are instances. Thus, what mathematics deals with are not empirical objects,[34] but concepts. A concept captures several real world objects but a concept is not a mere collection of objects. This is why Wittgenstein calls a concept 'the limit of the empirical.'[36] How does a concept differ from the instances of it? A concept does not cover an instance in entirety. When we say that 2 sheeps is an instance of the concept 2, we ignore such facts as that the sheep are alive, that they have tails etc. Secondly, a concept is permanently incomplete i.e., no amount of instances either exhausts or completely determines a concept.

According to Wittgenstein, mathematics grows by forming new concepts[37] or enlarging the old ones. Mankind is completely free to carry out this activity. That is why, he is averse to the contention of those who postulate a mathematical reality, that the reality could be extended in a particular direction and not in the other.[38]

V) Autonomy and Control : (a) A technique or a concept in mathematics is often enlarged in response to demands made upon it by the developments in other techniques and concepts. For example, the concept of integration was enlarged from Rieman's conception to that of Lebesgue's because the related concept of function was already extended to include inordinately discontinuous functions. Such expansion is mostly uncontroversial like the expansion which takes place in finding the solution of an external problem.

The introduction of the notion of the transfinite numbers was an autonomous development. It could be compared with the invention of the non-Euclidean Geometry. In both the cases, the growth took place not because the solution of an external problem demanded it, but because an inventive scientist made an ingenious extension of an existing concept or of several concepts. In inventing the transfinite numbers for example, Cantor extended the concepts of counting and number.

The question now is, how to distinguish a bonafide devel opment from ravishment of imagination ? Why should we accept the transfinite numbers as genuine mathematical entities? Is there an objective criterion to differentiate a natural growth from a cancerous one?[39] At different times philosophers had suggested different criteria.

(b) The intuitionists proposed that the basic intuitions of time and space are central to mathematics and that the invention of the transfinite is violative of the intuition of time. But intuitionist conception of mathematics is totally foreign to Wittgenstein's way of thinking. For him, mathematics is the product of a mathematical activity and that activity consists of making decisions according to a well established practice. There are not ingrained intuitions in mind which are revealed in mathematics. He remarks, "Intuitionism is all bosh–entirely."[40]

Russell had introduced the fear of a contradiction as a control on growth of mathematics. Acceptance of a new concept must not lead to a contradiction. There are two ways in which a contradiction controls growth. One, a new concept must be presented in the form of a proposition and the proposition requires a proof. A proof must not contain a contradiction. Since in this paper we are not going to consider Wittgenstein's thoughts on proof, we leave this approach here. The second is that the axioms must not contradict each other. If they do, any proposition is provable in the system. Wittgenstein ridicules this fear of a conttradiction. He says that if there is a contradiction we can just ignore it.[41]

(c) We have already seen that, according to Wittgenstein, a concept is never complete and we are free to enlarge a concept in any manner. Freedom of mankind to carry out the mathematical activity at its pleasure is admissible for Wittgenstein. On what ground, then, can he condemn the invention of the transfinite number?

Wittgenstein compares the case of the transfinite with a situation in chemistry. Suppose the valency technique allows the expression $H_2 O_4$. There is no substance in reality that answers to this expression.

What will be our reaction to this state of affairs. We will say, "That system of valencies wasn't chosen at random, but because it fitted well with facts. But once chosen, what is possible is what there is a picture of in the vallency language. We have adopted a language in which it makes sense to say $H_2 O_4$ .... It isn't true, but it makes sense."[42]

Now let us see the position of 'aleph-not'. True, it is not possible to reach aleph-not in ordinary arithmetic. But suppose it was possible. What sense the expression 'aleph-not' has? For this we have to imagine a situation where 'aleph-not' cannot be describeed without bringing in the use to which it is put.[43] This is the device of a language game.

In the context of ordinary arithmetic 'aleph-not' cannot be used in the sentence, 'Give me aleph-not rupees.' Not because there are not aleph-not rupees but because 'aleph-not' is not a number in the sense that 4 is a number. On the other hand, we can say, 'Jack knows aleph-not multiplications', in the sense that given two natural numbers he can find the product.[44]

The confusion resulting from using 'aleph-not' in the sense of 'four' conferred certain charm on 'aleph-not'. A calculus was created in which 'aleph-not' could be interpreted in this charming way. This is a retroactive revision. But this calculus is artificial and a-historical. Transfinite must not confer meaning upon the calculus, instead of getting one from it.[45]

This is the reason why Wittgenstein regards the transfinite as an arbitrary, aimless and senseless construction. It is not a natural growth of mathematics. As a mathematical theory, the transfinite and its calculus given by Russell may be consistent. But as a metaphysics of mathematics it is responsible for much sin.[46]

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Notes

1. See Monk (1990).
2. Dedekind (1963), p. 45.
3. Cantor (1955), p. 67
4. Wittgenstein (1983), p. 16. Also Monk (1990), pp. 415-16.
5. Rusell, B.(1959, p.78
6. Rusell B. (1914), p. 214.
7. Rusell B. (1930), p. 139-40.
8. *Ibid*, p. 93.
9. L 14.
10. Rusell (1959), p. 96.
11. *Ibid*, p.75.
12. *Ibid*, p. 61.
13. Monk (1990), p.440 Monk gives a quotation from R which is not found in my edition of 1956.
14. PI. Remark 241.
15. L 4. All the calculi in mathematics have been invented to suit experience and then made independent of experience.
16. R 160. A rule qua rule is detached ... although what gives it importance is the facts of daily experience.
17. L 75.
18. L 107 mathematical truth isn't ... it is the natural way to do it.'
19. L 159.
20. *Ibid*.
21. L 172. '... that logic should be, as one might say, is no way arbitrary.'
22. R 81.
23. L 172.
24. L 183-184.
25. L 271
26. L 153-54.
27. L 150.

28. L 94-95.

29. L 93.

30. L 124.

31. Russell (1959), p. 75.

32. R 133. It is the use outside mathematics and so the meaning of the signs that makes the sign game into mathematics.

33. Steve (Gerrard) (1991), p. 126.

34. R. 14.

35. By objects we do not mean 'things' but also processes and practices and groups of objects.

36. R. 121.

37. R 180. But he adds that there are regions in mathematics where the concept formation plays no part.

38. L 137.

39. For Wittgenstein the work of Cantor was 'a cancerous growth, seeming to have grown out of the normal body aimlessly and senselessly.' Quoted in Ray Monk (1990), p. 440.

40. L 237.

41. L 138. A contradiction does not even falsify anything. Let it lie. Do not go there.

42. L 146

43. Monk (1990) p, 330.

44. L 170.

45. R 63.

46. Monk (1990), p. 416.

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