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DEVIANT LOGICS FOR QUANTUM MECHANICS

' If, as one believes, all mathematics reduces to the mathematics of logic, and all physics reduces to mathematics, what alternative is there but for all physics, to reduce to the mathematics of logic ? '

Gravitation ; C. Misner, K. Thorne, and J. Wheeler.

1. Introduction : Birkoff and von Neumann [1] end their paper on ' The Logic of Quantum Mechanics ' by two questions, one of which is ' what experimental meaning can one attach to the meet and join of two given experimental propositions ? ' Since then a number of authors have followed the lead given by them and advanced the subject of quantum logic to a considerable extent. By now, as a result of this activity, quantum logic has attained a fair level of sophistication. But looking through the literature, we find that hardly anywhere one has tried to answer the above question within the scope of quantum mechanics, although attempts have been made to give operational meanings to the logical connectives. However, these attempts always seem to draw on the examples and experiences outside quantum mechanics, sometimes even outside physics. The notable contributions to the advance of quantum logic have been made, among others, by Feyerabend, Finkelstein, Giles, Putnam, Reichenbach, van Fraassen and von Weizsacker.

Till the time of Birkoff and von Neumann paper appeared, no one perhaps suspected that the advent of quantum mechanics as a science to deal with the microworld of molecules, atoms, nuclei and the like would bring about the change in the principles of classical logic. If one considers the laws to be consistent with the world of experience, one needs to conclude that the classical logic must be compatible with classical machanics which deals with the macroworld and indeed it is. Even though the rules of classical logic appear to have been laid down *a priori,* one strongly suspects that these rules were formulated by their authors with intuition based on everyday experience of the world around them. Hence, they seem to agree with classical physics, in general, with classical thought. The edifice of classical logic built by its authors is truly imposing. No one would have suspected that a crack would develop into this edifice as a result of the advent of quantum mechanics. Birkoff and von Neumann were the first to notice it. If the arrival of a new science required to study the microworld produced cracks in the edifice of (classical) logic, then the logic is not merely the ' creation of the human mind ' as it must have been thought of, but that it is empirical. Logic is determined by the picture of the world, the classical logic by that of the macroworld and quantum logic by that of the microworld.

In this paper we have attempted to present the logic of quantum mechanics by examining the measurement processes in quantum mechanics to determine the meanings of the logical connectives and to find whether they retain the meanings assigned to them in classical logic or they acquired new meanings in the light of strange new features of the quantum mechanical measurement processes. Most authors of quantum logic, particularly Finkelstein[2] and Putnam[3] have emphasized the fact that connectives in quantum logic retain classical meanings on account of operational definitions, though quantum logic differs from classical logic in only one respect. It is the failure of the distributive law: $\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ of the classical logic. We find in our investigations that in view of the peculiar nature of quantum mechanical measurements, the meanings of the logical connectives needed to conjoin simple experimental propositions to form compound statements need changes. It is seen that with the new meanings given to the logical connectives, this distributive law remains valid.

2. Experimental Propositions Of Quantum Mechanics : It is welknown that in classical mechanics the state of a mechanical system at any time is completely known if one specifies n generalized coordinates 'q' and n generalized momenta 'p' at that time, where n is the number of degrees of freedom of the mechanical system. These can be obtained by solving Hamilton's equations of motion with the initial conditions specified by the values of 'q' and 'p' at time t=o. Quantum mechanical state of microscopic systems is represented oy a certain wave–function ψ (or a state vector $|x>$) which can be expanded as a linear combination of eigenfunctions u_i (or eigenvectors $|i>$) of an operator representing an observable. The wave–function ψ (or the state vector $|x>$) is obtained by solving a differential equation known as Schrodinger equation which is of first order in time, the initial condition being the value of ψ at time t=o. In what follows we accept quantum mechanics in its conventional form as the correct representation of the microworld and accordingly build up our formalism.

Let an observable in quantum mechanics be denoted by A with the corresponding Hermitian operator denoted by the same letter A for convenience. We know from quantum mechanics that A must possess sufficient number of eigenstates, often infinite, that any state vector whatever can be expanded in terms of the corresponding eigen-vectors. The spectrum of the corresponding eigenvalues may be discrete (finite or denumerably infinite, degenerate or non-degenerate) or continuous. For our analysis we shall assume that the eigenvalue spectra of all the operators we shall introduce, be discrete, finite and non-degenerate. Let the eigenvectors of A be denoted by $|a_i>$ where $i=1, 2 \ldots, n$ $(<\infty)$. If A represented an observable, then any arbitrary state vector $|x>$ can be expanded as $|x> = \sum_{i=1}^{n} |a_i><a_i|x>$. If a_i denote the eigenvalues of A for the states $|a_i>$, then $A|a_i> = a_i|a_i>$. The process of measurement in quantum mechanics consists of three successive stages : (1) A preparatory stage when the quantum mechanical system S is 'prepared' to be in an arbitrary state $|x>$, considered as the initial state; (2) a working stage in which the 'prepared' system S interacts with the measuring apparatus (a macroscopic body), we shall call the analyzer and goes over to the superposition state $|x> = \sum_{i=1}^{n} |a_i><a_i|x>$ and (3) a registering stage in which the system S is registered in one of the eigenstates forming the superposition in the above expression (reduction of the wavepacket). Hence the process of measurement can be described as

$$\xrightarrow{(1)} |x> \xrightarrow{(2)} \sum_{i=1}^{n} |a_i><a_i|x> \xrightarrow{(3)} |a_i>$$

Experimental Proposition (EP) is a statement of the form : "The result of a measurement on a system S in the state $|x>$ and for the observable A is ". The completion of this sentence can be made in two ways. (1) " a real number a_i giving the value of the real dynamical variable (observable) A represented by the Hermitian operator A at time t" and (2)"...that the system, after the measurement is complete, is described by the state vector $|a_i>$". If the statement is completed in the manner (1), we shall call that statement as the experimental propositon of the first kind (EP I) and denote by, say, α_i. If the statement is completed in the menner (2), we shall call that statement as the experimental proposition of the second kind (EP II) and denote it by $\propto_i$'. Thus (EP I) is of the form M ($S_{A|x>}$)=a_i and (EP II) is of the form M($S_{A|x>}$)=$|a_i>$. In the present paper, we shall concern ourselves with EP I's only and drop the suffixes I and II.

3. *Experimental Meaning of Logical Connectives :*

In this section we shall consider the problem of introducing compound statements involving two or more EP's of the type described in the previous section and giving experimental meaning to the logical connectives conjoining them. We first consider the case of an observable represented by the Hermitian operator A with the eigen-value spectrum a_1, a_2 ....,a_n. $|a_1>$, $|a_2>$, , $|a_n>$ are the eigenvectors of A belonging to these eigenvalues. We assume to make measurement on identical copies of the system S prepared to be in the same state $|x>$ to measure the observable A. We denote, as in section 2, the EP's M ($S_{A|x>}$) = a_i as $\propto_i$. The aforementioned measurements will furnish the results a_1 or a_2 or a_3 or ..., not necessarily in that order, with probability measure associated with each of them. This means that a_1 or a_2 or ... are the possible (exclusive) results of the measurement process. If we put n = 2, then we may say that result of this measurement is either a_1 or a_2 but never both at the same time, when measurments are repeated on the identical samples in the same state. This measurement statement (for n = 2) can be translated into the logical language as $\propto_1 \vee \propto_2$, where the connective '$\vee$' is used in the *exclusive* sense. Let B be another observable with the corresponding Hermitian operator B with the complete set of eigenvectors $|b_i>$. This observable may or may not be compatible with the observable A. The measuiements on identical copies of S in the state $|x>$ to measure A and B not simultaneously, but successively, should give the results as a_i's or b_i's occuring singly at a time. Such measurement results can be put into the logical language as $\alpha_i \vee \beta_j$, with '$\vee$' again in the exclusive sense where the EP β_j is $M(S_{B|x>}) = b_j$. The results of successive measurements of one or more observables on identical copies of a quantum mechanical system, all prepared to be in a given state, are expressible in terms of an exclusive disjunction given by the truth-table 1.

Truth–table 1.

Having given the experimental meaning to disjunction, we now turn to the connective '$\wedge$' symbolizing conjunction. Let L and M be Hermitian operators corresponding to two observables. Let the discrete eigenvectors of these operators be denoted by $|l_i>$ and $|m_j>$ and the corresponding eigenvalues to which they belong be l_i and m_j. The indices i and j run through discrete integral values. Obviously $|l_i>$ and $|m_j>$ form a complete set of eigenvectors. We want to give meaning to the problem of *simultaneous* measurement of two observables. Suppose an experiment to measure simultaneously the values of L and M is performed on a system S in an arbitrary state $|x>$, we shall consider the result as read on two guages – L-guage and M-guage – *simultaneously*. We imagine this as some kind of coincidence measurement technique used with L–guage and M–guage expected to respond at the same time. If we denote the EP $M(S_{L|x>}) = l_i$ as λ_i and $M(S_{M|x>}) = m_j$ as μ_j, then the L–guage reading l_i and M–guage reading m_j gives the experimental proposition 'the results of the simultaneous mesurement of L and M on S in the state $|x>$ are the real numbers l_i and m_j. This EP can be translated into the the logical language as $\lambda_i \wedge \mu_j$. Hence the measurement of *two* observables furnishes us with the EP which is a conjunction of two EP's.

In studying conjunction we must distinguish between two cases : (1) when the two operators commute i. e. the corresponding observables are compatible, and (2) when the two operators do not commute and the corresponding observables are incompatible.

Case (1) : Let the two commuting operators be denoted by A and B, they satify [A, B] = O. We know from a well-known theorem in quantum mechanics that A and B have a *common* complete set of eigenvectors, hence we put $|a_i\rangle = |b_i\rangle = |a_i\ b_i\rangle$, where we have indicated the common set of eigenvectors as $|a_i\ b_i\rangle$ in which the letters a_i and b_i with the same subscript 'i' are inserted within the ket symbol $|\ \rangle$ sigifying that the ket $|a_i\ b_i\rangle$ is an eigenket of both A and B. Since A and B are observables, we have $|x\rangle = \sum_{i=1}^{n} |a_i\ b_i\rangle \langle a_i\ b_i\ |\ x\rangle$. A simultaneous measurement of A and B on the system S in the state $|x\rangle$ will furnish some a_i as the reading of the A-guage and some b_i of the B-guage simultaneously, where $A\ |\ a_j\ b_i\rangle = a_i\ |\ a_i\ b_i\rangle$ and $B\ |\ a_i\ b_i\rangle = b_i\ |\ a_i\ b_i\rangle$ (reduction of the wave-packet) . In the logical language, this is $\propto i \wedge \beta i$. We note that the index-set I, such that $i \in I$, is so arranged that the results of the above measurement are $a_i\ b_i$. This means that when A-guage reads a^i B-guage reads b_i . It never happens that the results are a^i and b_j with $i \neq j$, hence the statement $\propto i \wedge \beta j$ ($i \neq j$) is always false in this case. To construct the truth-table for conjunction '$\wedge$' in the case of compatible observables, we consider the experimental set-up with two coincidence guages to read the values of A and B. If the A-guage reads a_i and B-guage simultaneously reads b_i , then we assign the truth value T to both $\propto_i$ and β_i and $\propto_i \wedge \beta_i$ is also assigned the truth value T. If the A-guage registered reading a_i , but B-guage does not register anything, we shall assign the truth value T to $\propto_i$, F to β_i and F to $\propto_i \wedge \beta_i$. The reason is that by the very nature of the coincidence technique, this may be considered as the apparatus malfunction and the corresponding reading is rejected. So also if we assign F to $\propto_i$, T to β_i, then $\propto_i \wedge \beta_i$ has the truth value F.

Lastly, if both guages fail to respond, we assign F to $\propto_i$, F to β_i, and F to $\propto_i \wedge \beta_i$ leading to the truth–table 2.

Truth–table 2

We see that the above truth-table coincides with that of classical logic.

Case (2) : Let the two non-commuting Hermitian operators be denoted by P and Q. These correspond to two incompatible observables. Let P and Q satisfy $[P, Q] = iR$, where R is also a Hermitian operator corresponding to an observable. We denote the EP "the result of the measurment of P on S in the state $|x>$ is p_i (a real eigenvalue) as φ_i" and the EP "the result of the measurement of Q on S in the state $|x>$ is q_j (a real eigenvalue) as ψ_j". In these EP's the results of measurements of P and Q are supposed to have the *sharp* values p_i and q_j with the standard deviations $\triangle p$ and $\triangle q$ equal to zero. According to the uncetainty principle, if P and Q are measured simultaneously, $\triangle p \triangle q \geqslant \frac{r}{2}$, where r is the expectation value of R in state $|x>$. From this we see that φ_i is true if $\triangle p = 0$ and ψ_j is true if $\triangle q = 0$, but φ_i is false if $\triangle p \neq 0$ and ψ_j is false if $\triangle q \neq 0$ †. From the uncertainty principle, it is clear that $\varphi_i \wedge \psi_j$ is false only in the case when both φ_i and ψ_j are true, but it is true otherwise, resulting in the truth-table 3.

Truth–table 3

For the purpose of extending the analysis to more than two EP's conjoined by the disjunction '$\vee$', let us consider a Hermitian operator A corresponding to some observable. An EP of

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† In fact, for clarity, one may say that φ_i is true if $\triangle p < \frac{r}{2\triangle q}$ and ψ_j is true if $\triangle q < \frac{r}{2\triangle p}$, but φ_i is false if $\triangle p \geqslant \frac{r}{2\triangle q}$ and ψ_j is false if $\triangle q \geqslant \frac{r}{2\triangle p}$.

the form $M(S_{A|x}) = a_i$ is detoted by α_i, $i = 1, 2, 3, \ldots n(<\infty)$. We know from quantum mechanics that the system can be only in *one* state at a time. Hence at some time if $\alpha_i\,(1 \leqslant i \leqslant n)$ is true then all the other α_k's with $k \neq i$ are false at that time. We write $\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n$ to mean that the system S has the property described by the observable A. Similarly for another Hermitian operatar B with the spectrum of eigenvalues $b_1\, b_2, \ldots, b_m$ (in general $m \neq n$), an EP of the form $M[S_{B|x}] = b_j$ will be denoted by β_j. Using the same argument, we concider $\beta_1 \vee \beta_2 \vee \ldots$ $\ldots, \vee \beta_m$ as an always true statement to mean that the system S has the property described by the observable B. Clearly in the above expressions, we see that the disjunction is true if only one disjunct is true and others are false at a time, the disjunction is false if two or more of the disjuncts are true at a time. This is the requirement of an exclusive disjunction we have introduced in our quantum logic.

Regarding the connective '$\wedge$' signifying conjunction, we cannot extend its meanings to a statement of the form $\alpha \wedge \beta \wedge \gamma \wedge \ldots$ of more than two EP'S as this statement cannot be given a meaning, particularly for incompatible observables, since it refers to the result of simultaneous measurement of more than two observables. Even in quantum mechanics such measurements are hardly required and discussed. As the connective '$\wedge$' conjoins two EP'S which express the result of simultaneous measurement of two observable on a system S in an arbitrary state, we consider a statement of the form $\alpha_i \wedge \alpha_j$ as always false, where α_i and α_j are the EP's corresponding to the *same* observable. This is due to the fact that if α_i is true, α_j is false and that an observable is compatible with itself which follows the truth-table 2. However, the statement "the system has the properties described by the observables A and B at the same time" will be described by the expression $(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n) \wedge (\beta_1 \vee \beta_2 \vee \ldots \vee \beta_m)$ according to the meaning given to the connective

'$\wedge$'. If A and B are compatible observables, the expression $(\alpha_1 \vee \alpha_2 \vee \dots \vee \alpha_n) \wedge (\beta_1 \vee \beta_2 \vee \dots \vee \beta_n)$, where we have put m=n, is an always true statement. If P and Q are incompatible observables with the corresponding EP's denoted by φ and ψ then the statements $(\varphi_1 \vee \varphi_2 \vee \quad \vee \varphi_n)$ and $(\psi_1 \vee \psi_2 \vee \dots \vee \psi_m)$ are alwys true. but the statement

$$(\varphi_1 \vee \varphi_2 \vee \quad \varphi_n) \wedge (\psi_1 \wedge \psi_2 \vee \dots \vee \psi_m)$$

where, in general, $m \neq n$, signifying that the system has the properties P and Q at the *same time*, is an always false staement. It is often implied that a quantum mechanical systm S simultaneously possesses properties described by the incompatible observables P and Q at all times and that it is the measurement process that introduces the uncertainty. This is evidently false as the above result shows.

4. *The Distributive Law* : In the classical logic, we have two different forms of the distributive law. We shall state each forms for the two cases below. The generalization is obvious. One form is

(I) $\alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$ and

(I′) $(\alpha \wedge \beta) \vee (\gamma \wedge \delta) = (\alpha \vee \gamma) \wedge (\alpha \vee \delta) \wedge (\beta \vee \gamma) \wedge (\beta \vee \delta)$

The other form is

(II) $\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ and

(II′) $(\alpha \vee \beta) \wedge (\gamma \vee \delta) = (\alpha \wedge \gamma) \vee (\alpha \wedge \delta) \vee (\beta \wedge \gamma) \vee (\beta \wedge \delta)$.

First, let us consider the form (I). We apply it to two observables A and B with a complete spectrum of only one eigenvalue and two eigenvalues respectively. Then (I) can be written as $\alpha \vee (\beta_1 \wedge \beta_2) = (\alpha \vee \beta_1) \wedge (\alpha \vee \beta_2)$. In this experssion we note, on the right hand side, there occur expression of the form $(\alpha \vee \beta_1)$ and $(\alpha \vee \beta_2)$ conjoined by the connective '$\wedge$'. These expressions are obtained by conjoining two EP's corresponding to observables which may themselves be incompatible with other, so that it is impossible to say whether the EP's that correspond to $(\alpha \vee \beta_1)$ and $(\alpha \vee \beta_2)$ are compatible or incomatible with each other. Therefore, within the scope of definition of the connective '$\wedge$' which has different truth-tables for compatible and incompatible observables, these EP's cannot be conjoined together by '$\wedge$'. This means that this form of the distributive law lies outside the ambit of our logical system. In any case, this form has not even been mentioned by Finkelstein and Putnam. Similarly in the expression $(\alpha_1 \wedge \alpha_2) \vee (\beta_1 \wedge \beta_2) = (\alpha_1 \vee \beta_1) \wedge (\alpha_1 \vee \beta_2) \wedge (\alpha_2 \vee \beta_1) \wedge (\alpha_2 \vee \beta_2)$ the right hand side contains expressions $(\alpha_1 \vee \beta_1)$, $(\alpha_1 \vee \beta_2)$, . conjoined by the connective '$\wedge$'. This expression involves more than two EP's conjoined by the connective '$\wedge$' to which it is not possible to give an empirical meaning. Besides the EP's $(\alpha_1 \vee \beta_1)$, $(\alpha_1 \vee \beta_2)$, .. may correspond to neither compatible nor incompatible observables and thus cannot be meaningfully conjoined together by the connective '$\wedge$'. Hence we shall speak no more about this form of the distributive law in this paper.

Secondly, we consider the form (II) which is the form admissible in our formalism of quantum logic. Here we shall distinguish between two cases : Case (1) : The two observables A and B are compatible and the corresponding operators commute.

Case (2) : The two observables P and Q are incompatible and the corresponding operators do not commute.

We first consider case (1). In this case, we shall show that the distributive law holds when (a) the operator A has the complete spectrum of one eigenvalue and operator B has two, (b) A has the spectrum of 2 eigenvalues, B also has two and (c) A has the spectrum of 3 eigenvalues and B also has three. We shall show this by means of appropriate truth-tables.

(a) Let the EP : $M(S_{A|x>}) = a$ be denoted by α and the EP : $M(S_{B|x>}) = b_i$ by β_i, $i=1, 2$. In the case (a) we construct the following truth-table 4.

Truth – table 4

The truth values in the second and the seventh columns of the truth-table 4 show that the distributive law $\alpha \wedge (\beta_1 \vee \beta_2) = (\alpha \wedge \beta_1) \vee (\alpha \wedge \beta_2)$ holds.

We now consider the case (b), where the operator A has the spectrum of two eigenvalues and B has also two. In this case we denote the EP's corresponding to the observable A as α_1 and α_2 and the EP's corresponding to the observable B as β_1 and β_2. For this case, we have the following truth - table 5.

Truth–table 5

The truth values in the columns 4 and 12 in the truth-table 5 show that the distributive law holds. Finally, in the case (c) we denote the EP's corresponding to the observable A as α_1, α_2 and α_3 and the EP's corresponding to the observable B as β_1, β_2 and β_3. In this case we have the following truth-table 6.

Truth–table 6

The truth values in the columns 8 and 19 in the truth-table 6 show that the distributive law holds even in this case. It is possible to construct truth-tables for more complex cases involving more than three eigenvalues each and see that the distributive law holds in all these cases.

Secondly, we consider the other case of incompatible observable. Again, here we look into (a) P has only one eigenvalue and Q has two, (b) P has 2 eigenvalues, Q has also two and (c) P has 2 eigenvalues but Q has three. Let the EP : $M(S_{P|x>}) = p$ be denoted by φ and the EP : $M(S_{Q|x>}) = q$

by ψ. In the case (a) we denote EP corresponding to the observable P as φ and the EP's corresponding to the observable Q as ψ_1 and ψ_2. When we construct the truth-table 7 for this caes, we find that the truth-table has the following form.

Truth - table 7

Here we have a surprise ! The distributive law has failed ! But looking closer, we see that this is expected as the observable P, which has only one eigenvalue, is a constant like the electron charge. Hence the operator P corresponding to this observable is just a number which commutes with every other operator, that is, the observable P is compatible with every other observable. We make a mistake in the fiirst place in taking P with one eigenvalue incompatible with Q.

In the case (b) we denote the EP's corresponding to P as φ_1 and φ_2 and the EP's corresponding to Q as ψ_1 and ψ_2. In this case we are led to the following truth-table 8 .

Truth-table 8

The truth valus in the columns 2 and 8 indicate that the distributive law holds. Finally, in the case (c) we denote the EP's corresponding to P as φ_1 and φ_2 and the EP's corresponding to Q as ψ_1, ψ_2 and ψ_3 (note that $m \neq n$ here). We construct the following truth-table 9 .

Truth–table 9

In this case as well, we see that the distributive law holds. One can easily verify this for more complex cases. From the above truth-tables, we see that the distributive law holds for EP's both in the case of compatible and incompatible observables. The apparant failure of the law in the case (2a) is actually the confirmation of its validity.

5. Connectives of Negation, Implication and Equivalence : It can be seen from the above analysis that the distributive law in the from

$(\lambda_1 \vee \lambda_2 \vee \ldots) \wedge (\mu_1 \vee \mu_2 \vee \ldots) = (\lambda_1 \wedge \mu_1) \vee (\lambda_1 \wedge \mu_2) \vee \ldots \vee (\lambda_i \wedge \mu_j) \vee \ldots$ holds for the EP's corresponding to both the compatible and incompatible observables. We have seen that this from, but not the other, is meaningful in our empirical logic within the scope of meaning we have assigned to the connectives of disjunction '$\vee$' and of conjunction '$\wedge$'. Another connective we have not mentioned so far and which has a basic function in further development of our logical system is negation '$\neg$'. If we denote the EP : $M(S_{L|x>}) = l_i$ as λ_i, then we shall denote the EP : $M(S_{L|x>}) = l_j,\ j \neq i$ as $\neg\lambda_i$. We see that the meaning given to the connective of negation is quasi-classical.

Now that we have given empirical meanings to the connectives '$\vee$', '$\wedge$' and '$\neg$' we set to define some additional connectives like those of implication and equivalence for both campatible and incompatible observables. Our definition of implication happens to provide the same truth-table for it, in these cases as in the classical case : a conditional with a true antecedent and a false consequent is false. This requires that in the case of compatible observables we put $(\alpha \supset \beta) \equiv (\neg\alpha \vee \beta) \vee (\neg\alpha \wedge \beta)$ where α and β are the EP's corresonpding to the compatible observables A and B. We illustrate this by the following truth-table 10, which can easily be constructed using the above expression on the right.

Truth-table 10

In the case of incomptible observables, we have

$(\varphi \supset \psi) \equiv (\neg\varphi \vee \psi) \vee \neg(\neg\varphi \wedge \psi)$

where φ and ψ are the EP's corresponding to the incompatible observables P and Q. This also leads to the same truth-table 10 given above. In the above two expressions we have already used the connective '$\equiv$' which is known as that of equivalence. The concept of equivalence in the sense that two EP's (occuring on either side of that connective) are materially equivalent when they have the same truth-value ought to be accepted in quantum logic regardless of the nature of EP's as to whether they correspond to compatibles. This requires that if λ and μ are two EP's corresponding to two observables either compatible or incompatible with each other, then we put the equivalence relation between them as $(\lambda \equiv \mu) \equiv (\lambda \supset \mu) \vee \neg(\mu \supset \lambda)$. This holds in both cases. However, in the case of compatible observables with the classical meaning of '$\wedge$', one may also put $(\alpha \equiv \beta) \equiv (\alpha \supset \beta) \wedge (\beta \supset \alpha)$.

Finally, we present a few consequences of the empirical meanings given to the set of connectives '$\vee$' '$\wedge$' and '$\neg$'. Since experimental meaning have already been assigned to these connectives, one need not strive to seek empirical meanings to other connectives defined in terms of these, such as implication and equivalence. Yet it is fruitful to briefly investigate the following questions : (1) For which pair of EP's λ and μ, the presence of one, say λ, entails the presence of the other? (2) For which pair of EP's λ and μ, the presence of one is entailed by the other? Both questions can be answered within the framework of our logical system, provided we distinguish between two cases of compatible and incompatible observables. (1) If α_i and β_i are the EP's corresponding to two compatible observables **A** and **B**, we assert that the presence of α_i (or β_i) entails $(\alpha_i \vee \beta_i) \vee (\alpha_i \wedge \beta_i)$. Logically this means that $\alpha_i \supset [(\alpha_i \vee \beta_i) \vee (\alpha_i \wedge \beta_i)]$ is a tautotogy which indeed it is. Empirically it would mean that if α_i is true (or if β_i is true) then $[(\alpha_i \vee \beta_i) \vee (\alpha_i \wedge \beta_i)]$ is true also, but the converse is not true. This can be seen from the following truth-table 11.

Truth-table 11

The above discussion shows that when α_i is true (or when β_i is true), then $\alpha_i \vee \beta_i$ (signifying *successive* measurements of A and B) and $\alpha_i \wedge \beta_i$ (which signifies *simultaneous* measurements of A and B) are two mutually exclusive EP's. Similarly if if φ_i and ψ_j are the EP's corresponding to two incompatible observables P and Q ,we assert that the presence of φ_i (or ψ_j) entails the presence of $(\varphi_i \vee \psi_j) \vee \neg (\varphi_i \wedge \psi_j)$. It is easy to verify that $\varphi_i \supset [(\varphi_i \vee \psi_j) \vee \neg (\varphi_i \wedge \psi_j)]$ is a tautology and that if φ_i is true (or if ψ_j is true), then $[(\varphi_i \vee \psi_j) \vee \neg (\varphi_i \wedge \psi_i)]$ is true also. Notice the presence of $\neg (\varphi_i \wedge \psi_j)$ in the case of incompatible observables instead of $(\varphi_i \wedge \psi_j)$. Hence when φ_i is true (or when ψ_j is true), then $\varphi_i \vee \psi_j$ (signifying *successive* measurements of P and Q) and $\neg (\varphi_i \wedge \psi_j)$ (signifying *simultaneous* measurement of P and Q) are two mutually exclusive EP's.

(2) For compatible observables, one easily verifies that α_i (Or β_i) is entailed by ($\alpha_i \wedge \beta_i$). This means that if $(\alpha_i \wedge \beta_i)$ is true then α_i as well as β_i is true and that $(\alpha_i \wedge \beta_i) \supset \alpha_i$ (or β_i) is a tautology. In a similar fashion, one finds that φ_i (or ψ_j) is entailed by $\neg (\varphi_i \wedge \psi_j)$, where φ_i and ψ_j are the EP's for two incompatible observables.

6. *Concluding remarks :* We have shown in the paper that the basis of logical statements, as is often claimed, is not just *a priori*, but has to take into account empirical truths. A logic originated and developed during the classical period cannot but be at variance with empirical truths of quantum mechanics developed to explain the behaviour of the microworld. If logic is claimed to be empirical, is it merely dropping a well-known law of classical logic and nothing else ? We answer this question by saying that it is much more. If the logic is empirical, then one must investigate the empirical meanings of the

basic connectives like those of disnjunction and conjuction in the light of new insights into measurement processes of the microworld. Do the classical meanings of these connectives, which have been time-tested in the classical context, remain unscathed when we look beyond the classical world and peep into the microworld of atoms, nuclei and elementary particles or do they undergo any change ? Our investigation has shown that we have to adopt the latter alternative. In this process we find that we have to distinguish between compatible and incompatible observables of quantum mechanics and adopt two different deviant logics for these cases.

APPENDIX I

In this appendix we present two truth-tables 12 and 13 for EP's corresponding to compatible and incompatible observables respectively and list the laws in formal logic which hold and which do not hold in our deviant logics.

Truth-table 12

In the above table, the disjunction '$\vee$' is an exclusive disjunction. The negation and conjuncion happen to coincide with their classical counterparts, so do the implication and equivalence. We have expressed the implication in terms of the above connectives of negation, disjunction and conjuction as $(\propto \supset \beta) \equiv (\neg\propto \vee \beta) \vee (\neg\propto \wedge \beta)$. We express the eqivalence $(\propto \equiv \beta)$ as either $(\propto \supset \beta) \vee \neg (\beta \supset \propto)$ or $(\propto \supset \beta) \wedge (\beta \supset \propto)$.

Using the above truth-table 12, one may easily verify the following :

(1) $(\propto \vee \neg\propto)$ is at autology, (2) $(\propto \wedge \neg\propto)$ is a contradiction, (3) Modus Ponens holds, (4) Modus Tollens holds, (5) Disjunctive Syllogism holds, (6) Hypothetical Syllogism holds, (7) Addition fails, (8) Simplification holds, (9) Conjunctiou

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tible observables can be expressed in terms of '$\neg$', '$\vee_{cl}$' and '$\wedge_{cl}$' as follows :

$$\neg\varphi \vee (\psi \wedge x)$$
$$\equiv [\neg\varphi \vee_{cl} (\psi \wedge x)] \wedge_{cl} \neg [\neg\varphi \wedge_{cl} (\psi \wedge x)]$$
$$\equiv [\neg\varphi \vee_{cl} \neg (\psi \wedge_{cl} x)] \wedge_{cl} \neg [\neg\varphi \wedge_{cl} \neg (\psi \wedge_{cl} x)]$$

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TRUTH – TABLES

Truth–table 1

α	β	$\alpha \vee \beta$
T	T	F
T	F	T
F	T	T
F	F	F

Truth–table 2

α_i	β_i	$\alpha_i \wedge \beta_i$
T	T	T
T	F	F
F	T	F
F	F	F

Truth–table 3

φ_i	ψ_j	$\varphi_i \wedge \psi_j$
T	T	F
T	F	T
F	T	T
F	F	T

Truth–table 4

$\propto$	$\wedge$	$(\beta_1$	$\vee$	$\beta_2)$	$(\propto \wedge \beta_1)$	$\vee$	$\propto \wedge \beta_2$
T	T	T	T	F	T	T	F
F	F	T	T	F	F	F	F
T	T	F	T	T	F	T	T
F	F	F	T	T	F	F	F

Truth–table 5

$(\propto_1$	$\vee$	$\propto_2)$	$\wedge$	$(\beta_1$	$\vee$	$\beta_2)$	$\propto_1 \wedge \beta_1$	$\propto_1 \wedge \beta_2$	$\propto_2 \wedge \beta_1$	$\propto_2 \wedge \beta_2$	$\vee$
T	T	F	T	T	T	F	T	F	F	F	T
T	T	F	T	F	T	T	F	T	F	F	T
F	T	T	T	T	T	F	F	F	T	F	T
F	T	T	T	F	T	T	F	F	F	T	T

Truth – table 6

α_1	α_2	α_3	β_1	β_2	β_3	$\alpha_1 \vee \alpha_2 \vee \alpha_3$	$\wedge$	$\beta_1 \vee \beta_2 \vee \beta_3$	$\alpha_1 \wedge \beta_1$	$\alpha_1 \wedge \beta_2$
T	F	F	T	F	F	T	T	T	T	F
T	F	F	F	T	F	T	T	T	F	T
T	F	F	F	F	T	T	T	T	F	F
F	T	F	T	F	F	T	T	T	F	F
F	T	F	F	T	F	T	T	T	F	F
F	T	F	F	F	T	T	T	T	F	F
F	F	T	T	F	F	T	T	T	F	F
F	F	T	F	T	F	T	T	T	F	F
F	F	T	F	F	T	T	T	T	F	F

$\alpha_1 \wedge \beta_3$	$\alpha_2 \wedge \beta_1$	$\alpha_2 \wedge \beta_2$	$\alpha_2 \wedge \beta_3$	$\alpha_3 \wedge \beta_1$	$\alpha_3 \wedge \beta_2$	$\alpha_3 \wedge \beta_3$	$\vee$
F	F	F	F	F	F	F	T
F	F	F	F	F	F	F	T
T	F	F	F	F	F	F	T
F	T	F	F	F	F	F	T
F	F	T	F	F	F	F	T
F	F	F	T	F	F	F	T
F	F	F	F	T	F	F	T
F	F	F	F	F	T	F	T
F	F	F	F	F	F	T	T

Truth – table 7

φ	$\wedge$	$(\psi_1$	$\wedge$	$\psi_2)$	$(\varphi \wedge \psi_1)$	$(\varphi \wedge \psi_2)$	$\vee$
T	F	T	T	F	F	T	T
F	T	T	T	F	T	T	F
T	F	F	T	T	T	F	T
F	T	F	T	T	T	T	F

Truth – table 8

$(\varphi_1$	$\vee$	$\varphi_2)$	$\wedge$	ψ_1	$\vee$	$\psi_2)$	$\varphi_1 \wedge \psi_1$	$\varphi_1 \wedge \psi_2$	$\varphi_2 \wedge \psi_1$	$\varphi_2 \wedge \psi_2$	$\vee$
T	T	F	F	T	T	F	F	T	T	T	F
T	T	F	F	F	T	T	T	F	T	T	F
F	T	T	F	T	T	F	T	T	F	T	F
F	T	T	F	F	T	T	T	T	T	F	F

Truth–table 9

φ_1	φ_2	ψ_1	ψ_2	ψ_3	$\varphi_1 \vee \varphi_2$	$\psi_1 \vee \psi_2 \vee \psi_3$	$\wedge$	$\varphi_1 \wedge \psi_1$	$\varphi_1 \wedge \psi_2$	$\varphi_1 \wedge \psi_3$	$\varphi_2 \wedge \psi_1$	$\varphi_2 \wedge \psi_2$	$\varphi_2 \wedge \psi_3$	$\vee$
T	F	T	F	F	T	T	F	F	T	T	T	T	T	F
F	T	T	F	F	T	T	F	T	T	T	F	T	T	F
T	F	F	T	F	T	T	F	T	F	T	T	T	T	F
F	T	F	T	F	T	T	F	T	T	T	T	F	T	F
T	F	F	F	T	T	T	F	T	T	F	T	T	T	F
F	T	F	F	T	T	T	F	T	T	T	T	T	F	F

Truth-table 10

α	β	$\alpha \supset \beta$
T	T	T
T	F	F
F	T	T
F	F	T

Truth-table 11

α_i	β_i	$\alpha_i \vee \beta_i$	$\alpha_i \wedge \beta_i$	$(\alpha_i \vee \beta_i) \vee (\alpha_i \wedge \beta_i)$
T	T	F	T	T
T	F	T	F	T
F	T	T	F	T

Truth-table 12

α	β	$\neg\alpha$	$\neg\beta$	$\alpha \vee \beta$	$\alpha \wedge \beta$	$\alpha \supset \beta$	$\beta \supset \alpha$	$\alpha \equiv \beta$
T	T	F	F	F	T	T	T	T
T	F	F	T	T	F	F	T	F
F	T	T	F	T	F	T	F	F
F	F	T	T	F	F	T	T	T

Truth-table 13

φ	ψ	$\neg\varphi$	$\neg\psi$	$\varphi \vee \psi$	$\varphi \wedge \psi$	$\varphi \supset \psi$	$\psi \supset \varphi$	$\varphi \equiv \psi$
T	T	F	F	F	F	T	T	T
T	F	F	T	T	T	F	T	F
F	T	T	F	T	T	T	F	F
F	F	T	T	F	T	T	T	T