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A NOTE ON THE TWO METHODS OF PROOF

Abstract : Two methods of proof, one direct and the other indirect, namely, Mathematical Induction and Reductio ad Absurdum are formalized in the language of Symbolic Logic. The purpose of formalization is to show the link between them. We have shown this link by restricting the scope of the arguments to the set N of natural numbers.

1. *Introduction*: In the development of mathematics as a science of abstract forms and structures over the decades of the past century and the present, the Greek concept of rigorous thinking which is the same as the idea of proof played a central role. In the process of this formalized deductive system, the discernment of a statement, called as a lemma or a theorem, form the definitions of *primitive terms*. The unproved statements known as *postulates* or *axioms* containing these terms required a 'proof' which asserts that a lemma or a theorem is logically implied by the postulates or axioms. The end of the arguments giving the proof usually ends with a statement like 'This completes the proof of the theorem'. It is clear that a mathematical system cannot be built by 'defining' the elements of the systems, the relations among these elements and the operations performed upon them in terms of other elements, the relations among them and the operations performed upon them ad infinitum. At some stage, one has to cut short these backward journey or face circulatory at some point. As circularity is not a desirable feature of mathematical discourse, one chooses to call halt at a stage when one arrives at the so-called primitive terms, statements and operations upon them which are considered the starting points where the primitive terms and statements are left unproved. These statements are postulates or axioms.

In the present note, we are strictly concerned with the notion of natural numbers and the methods of proof based on them. It is well-known that the method of Mathematical Induction is based on the concept of natural numbers. However, in our investigation we include another method, namely, Reductio ad Absurdum, which we present in such a form that it is made to depend on the notion of natural numbers. Though it may be objected that the method of Reductio ad Absurdum is far more general in its scope than it would appear from our restricted presentation, we confine ourselves to this very limited case to be able to show its close resemblance to the method of Mathematical Induction.

Through this note, we pay tribute to Gottlob Frege,[1] the Mathematician-Logician who held the view that the true statements of 'arithmatic' (the theories of natural and real numbers) are analytic by which he meant that they are logically deducible from purely logical laws with the help of arithmatical notions tacitly assuming that every true arithmatical statement is provable. His researches on the structures of natural numbers led him to derive all the propositions that have been subsequently known as Peano's axioms for the natural numbers. He held the view that the principle of mathematical induction, far from being a special principle of inference peculiar to the theory of natural numbers whose validity is apprehended by mathematical intuition, is part of the definition of 'natural number'. The natural numbers are to be defined as just those objects for which the mathematical induction is valid. We assert that Frege's ideas on mathematical induction can be extended to another principle, equally widely prevalent, namely, reductio ad absurdum in the restricted sense we envisage. To bring them both into one single category we intend to present them in the same language, that of *Symbolic Logic*, in agreement with Frege.

2. *The Two Methods of Proof*: The natural number system, as against the real number system, has an intuitive simplicity, lacking in most other mathematical systems as

this system constitutes the numbers used for counting since thousands of years of human civilization. The natural numbers have often been extensively handled over a long period of time without producing any known inner contradictions.[2] It is possible to arrive at the real number systems, by definition, from a postulate set for the much simpler and the more basic system of natural numbers. As a matter of fact the consistency of the great bulk of mathematics depends upon that of the very fundamental system of natural numbers, as seen from the researches of Frege, Peano, Dedekind and Cantor. The accomplishment of those mathematicians and the results they obtained has given a considerable feeling of confidence concerning the consistency of most of mathematics. We shall not state the full postulates set for the system of natural numbers as we shall have no occasion to make use, in our exposition, of all the postulates in the set. Nevertheless, we state one of them. It reads: If N is a set of natural numbers such that (1) N contains the natural number 1, (2) N contains the natural number k + 1 whenever it contains the natural number k, then N contains all the natural numbers. In the above we have used binary operation of addition denoted by +. This binary operation on the set of natural numbers is contained in other postulates of the natural number system which we assume to have been understood.

The above postulate is known as the postulate of finite induction. It is an extremely interesting postulate as it leads to, inter alia, a very important method of proof in mathematics known the 'principle' of mathematical induction. Often the 'principle' of mathematical induction makes its appearance as the 'method' of mathematical induction used widely in supplying proof to a number of theorems in mathematics, the binomial theorem being one of them. Mathematical induction is presented in the literature as a theorem, hence called a principle, to be proved from the above postulates of the natural number system. However, we shall be interested in the form of it as a method of proof, the principle underlying it having been proved already. We state the principle of mathematical induction as: Let P(n) be a proposition that is defined for every natural number n. If P(n) is true for some $n \in N$, and if, for each natural number k, P(k + 1) is true whenever P(k) is true for all natural numbers k, then P(n) is true for all natural numbers n.

This is the conventional form of the principle of which we shall present a formalized version in the next section. It is interesting to note that the principle of mathematical induction can be extended to apply to a case in which the natural number system is replaced by a denumerably infinite set of numbers. This is hardly surprising as such a set can always be put into a one-to-one correspondence with the set of natural numbers.

On the other hand, the method of reductio ad absurdum rests on two cardinal principles of classical logic, namely, the law of contradiction and the law of excluded middle. The former can be stated, somewhat loosely, that if P is any statement, then P and its denial $\neg P$ cannot both hold, that is, $\neg P \wedge P$ is false. The latter is stated as 'either P or some denial of P (i.e. $\neg P$) must hold, that is, there is no third or *middle* possibility. Putting this logically we have $P \vee \neg P$ to be true. In the above we have used the connectives $\neg$, $\vee$ and $\wedge$ of classical two-valued logic with the usual meanings. Conventionally what it means is illustrated by the following argument. Let P be the statement of any proposition to be established by the method of reductio ad absurdum. By this method we set about to show that any denial of P implies a denial of some previously assumed or established statement R. Now, by the law of contradiction, R and its denial cannot both be true. Since R is true, the denial of R is false. Since a true statement can never imply a false one, it follows that any denial of P must be false. But, by the law of excluded middle, either P is true or some denial of p is true. Since the denial is shown to be false, it follows that the proposition is true and hence our proposition is established.

We wish to achieve our aim of expressing these methods of proof in terms of formal logical parlance and then showing explicitly the correlation between them, by formalizing their statements in a suitable fashion using the language of classical symbolic logic.[3] Operating with the standard rules of inference and quantifiers, we shall show how this correlation can be established. We set the assumptions (A_1) and (A_2) for the mathematical induction and $(A_1)'$ and $(A_2)'$ for reductio ad absurdum in such a way that the comparison of their logical forms clearly shows the link between them, particularly *when the natural numbers are involved in both these methods.*

3. *Formalization*: We shall first outline the method of proof by mathematical induction. The principle of mathematical induction is an important property of the natural numbers (positive integers). It is particularly useful in proving propositions P(n) involving all positive integers when it is known that the propositions P(n) are valid for n = 1, 2, 3, 4 and it is conjectured that they hold for all positive integers.

The method of proof consists of the following steps:

1. Prove the proposition P(n) for some $y \varepsilon N$.
2. Assume the proposition P(n) to be true for n = k, where k is any positive integer.
3. From the assumption in 2. above, prove that the proposition P(n) is true for n = k + 1.
4. Since the proposition P(n) is true for n = 1 (step 1) it must (step 3) be true for n = 1 + 1 = 2 and from this for n =2 + 1 = 3 etc., and so must be true for all positive integers.

These four steps can be combined in one principle known as the principle of mathematical induction: Let there be associated with each positive integer n, a proposition P(n) which is either true or false. If firstly, $P(1)$ is true, and secondly if for all k, $P(k)$ implies $P(k+1)$ then $P(n)$ is true for all positive integers.

In the following, we present the principle of mathematical induction as above, in the language of formal symbolic logic. As made clear in section 2, the universe of discourse considered for the principle of mathematical induction is the set $N = \{1, 2, 3 \ldots\}$ of all natural numbers.

The principle of mathematical induction is formalized by stating the following two axioms:

(A1) $(\exists_y)\ (P(y))$.

(A2) $(k)\ [P(k) \supset P(k+1)]\ .\ k \in N$.

and prove the validity of $P(y)$, for all $y \varepsilon N$.

Proof :	1. $[P(y) \supset P(y+1)]$	by (A2) and Existential Instantiation (E. I.)
	2. $\neg P(y) \vee P(y+1)$	Material Implication (M. I.) and 1.
	3. $P(y+1)$	by (A1) and Disjunctive Syllogism (D. S.)
	4. $[P(y+1) \supset P(y+2)]$	by (A2) and E. I.
	5. $\neg P(y+1) \vee P(y+2)$	M. I. and 4.
	6. $P(y+2)$	by 3. and D. S.

Continuing the above sequence of steps for $y+3$, $y+4$, ..., we see that in general, $P(y+n)$ is true for all $n \varepsilon N$. Therefore $P(y)$, for all $y \varepsilon N$, is true.

We now outline the indirect method of proof, reductio ad absurdum. This method is particularly useful when the method of direct proof is not easily attainable. In order to to prove the truth of the proposition $P(n)$ for all $n \varepsilon N$ the method of reductio ad absurdum consists of the following steps.

1. Assume that $P(y)$ is false for some $y \varepsilon N$ i.e. $P(y)$ is false.

2. Assume that the inference of the negation of the proposition P(y) leading to the negation of the proposition P(y + 1) is false.
3. By a series of steps 1. and 2. above, we arrive at the result of P(y) is true.
4. By conjunction of the two statements 1. and 3., we arrive at the logical contradiction that P(y) is false and P(y) is true. Hence rejecting the assumption in 1., that (Py) is false, we prove that P(y) is true for all $y \in N$.

These four steps can be combined as follows: The indirect method of proof of reductio ad absurdum starts with the assumption that the proposition P(n) to be proved is false for some $y \in N$ and we arrive at the conclusion that P(y) is both true and false. This leads to the rejection of the assumption that the proposition P(n) is false for some $n \in N$. Therefore we established that P(n) is true for all $n \in N$.

In the following, we present the indirect method of proof, reductio ad absurdum, as outlined above in formal symbolic logic. Once again, as in the case of mathematical induction, the universe of discourse will be the set $N = \{1, 2, 3, \ldots\}$

We provide the following two axioms in the formalization of the method of reductio ad absurdum:

(A1)' $(\exists_y)(\neg P(y))$

(A2)' $(k)\{\neg[\neg P(k) \supset \neg P(k+1)]\}$

and prove the validity of P(y) for all $y \in N$.

Proof :	1. $\neg[\neg P(y) \supset \neg P(y+1)]$	by (A2)' and E. I.
	2. $\neg[\neg\neg P(y) \vee \neg P(y+1)]$	by 1. and M. I.
	3. $\neg[P(y) \vee \neg P(y+1)]$	by 2. and double negation (D. N.)
	4. $\neg P(y) \quad \neg\neg P(y+1)$	by 3. and De Morgan Law.

5.	$\neg P(y) \wedge P(y+1)$	by 4. and D. N.
6.	$P(y+1)$	by 5. and by simplification
7.	$\neg [\neg P(y+1) \supset \neg P(y+2)]$	by (A2)' and E. I.
8.	$\neg [\neg\neg P(y+1) \vee \neg P(y+2)]$	by 7. and M. I.
9.	$\neg [P(y+1) \vee \neg P(y+2)]$	by 8. and D. N.
10.	$\neg P(y+1) \wedge P(y+2)$	by 9. and De Morgan Law
11.	$\neg P(y+1)$	by 10. and simplification.
12.	$P(y+1) \wedge \neg P(y+1)$	by 6. 11 and conjunction.

Here the step 12. is an explicit contradiction in classical 2-valued symbolic logic. Hence the assumption (A1)' is false, i.e.

(i) $(\exists_y)(\neg P(y))$ is false,

(ii) $\therefore \neg [(\exists_y)(\neg P(y))]$ is true,

(iii) $(y)(\neg(\neg P(y)))$ is true by Quantifier Negation (Q. N.)

(iv) $(y)(\neg\neg P(y))$ is true

(v) $(y)(P(y))$ is true by D. N.

Therefore we have established that the proposition $P(n)$ is true for all $n \varepsilon N$.

4. *Conclusion*: In this paper, we have shown the link between two apparently unconnected methods of proof used frequently in mathematics, namely, mathematical induction and reductio ad absurdum. This link has been established through the formalization of these methods using the language of two-valued classical symbolic logic. In the process of showing this link the logical connective of negation $\neg$ has played an important role.

Indeed, the direct method of proof, mathematical induction, started with an initial affirmation of a proposition

$P(n)$, for all $n \varepsilon N$, and also of the law of inference by which the proposition $P(n+1)$ follows from $P(n)$ for all $n \varepsilon N$. However, in the indirect method of reductio ad absurdum we not only denied the proposition $P(n)$ for only one instance $P(y)$, $y \varepsilon N$ but also amended the law of inference itself.

It is interesting to note that both these methods finally establish the truth of the proposition $P(n)$ for all $n \varepsilon N$. However, the authors suggest that this is due to the use of diametrical negation in classical 2-valued symbolic logic.

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