PARADOXICAL TRUTH CONDITION OF MATERIAL IMPLICATION

I

In the following pages I shall reconsider the familiar problem that has arisen out of identification or alleged identification of material implication with the relation of inferribility. My purpose is to reach some clarification of the problem[1] and, if possible, some solution without abandoning the policy of not admitting any non-truth-functional concept.[2]

Russell has said that the theory of (material) implication is " the theory of how one proposition can be inferred from another."[3] Thus he evidently and consciously identifies the relation of material implication with the relation of inferribility. And here the critics hold that this identification leads to paradoxical results in view of the definition of material implication viz $p \supset q = df. \sim pvq$. And on this ground the critics hold that material implication has no serious claim to the name implication, that it provides no proper account of the relation of inferribility.

Doctrine of material implication is often sought to be defended against this charge by pointing out that this is a techincal notion which has effected tremendous convenience, that paradoxes are the truths of the system, that there are no paradoxes at all and so on. But it is still held " by a small but persistent group of logicians that the Standard Logic criterion of validity is too permissive as a test of correctness in the intuitive sense."[4]

I shall first show the necessity for including, as part of the meaning of material implication, the notorious paradoxical truth condition. Unless we can convince the critics of such necessity they cannot be pursuaded to accept the notion simply on the ground that paradoxes are harmless. For paradoxes, even if harmless, should not be admitted into a system if they are avoidable.

II

Any adequate theory of inference must provide room for distinguishing genuine inferences from what is not genuine inference. In addition to this every adequate theory of inference must provide room for distinguishing, within the class of genuine inferences, one inference from another. In other words ability to distinguish different inferences is one of the criteria of adequacy for any theory of inference.

To begin with, there is an intuitive difference between inferring q from p and inferring p from q; that is to say, it is intuitively clear that logically correct inference of q from p is different from logically correct inference of p from q. This intuition is reflected in the formal system by the fact that law of commutation is not admitted to hold of implication though it holds of such other binary connectives as conjunction, disjunction or equivalence. Any way there are different inferences like inference of q from p, inference of p from q, inference of $\sim$p from $\sim$q and so on.

Intuitive non-reversibility of inferential relation or non-derivability of '$p \supset q = q \supset p$', (or even '$(p \supset q) \supset (q \supset p)$') as a theorem in a system like PM takes the inference of q from p to be different from the inference of p from q. But how to distinguish them within the limits of a truth functional logic ? We must admit that there may hold a relation between p and q on the basis of which q can be validly inferred from p, but p cannot be validly inferred from q. Since the relation which permits q to be validly deduced from p is the inferential relation, the latter should be such that when it obtains between p, and q, q can be validly deduced from p, and not p from q. Thus if inference of q from p is to be distinguished from the inference of p from q we must so characterize the inferential relation as to make it non-reversible. And we think the doctrine of material implication has succeeded in doing this.

Thus suppose we take the expression 'p materially implies q' to be synonymous with the expression 'q is validly deducible from p', we can reformulate our question thus : How 'p materially implies q' be distinguished from 'q materially implies p' ? Here again the distinction we demand is not any sort of intentional or just typographical distinction. To be able to distinguish in the sense required it will have to be shown that '$p \supset q$' may be true but '$q \supset p$' may be false. That is there should be at least one condition under which '$p \supset q$' becomes true *and* 'q implies p' becomes false. In other words at least one of the truth conditions of ' p ⊃ q ' must *at the same time* be the falsity condition of ' q ⊃ p '. Now the conditions TT and FF are the common truth conditions of both ' p ⊃ q ' and ' q ⊃ p '. Therefore the only way of distinguishing the two inferences is to admit among the truth conditions of ' p ⊃ q ', the condition FT.[5] And this condition is alleged to be the paradoxical truth condition of ' p ⊃ q '.

I, therefore, think that ' ⊃ ' be so defined as to make ' p ⊃ q ' true under condition of FT. For otherwise the distinction between ' p ⊃ q ' and q ⊃ p ' cannot be maintained. And the obliteration of the distinction will not only be contra-intuitive but also formally disastrous. If FT condition is dropped along with TF condition then whenever the other two conditions hold we would be justified in inferring q from p as also p from q, even when p and q are not equivalent. And surely p can imply q without being equivalent with q or otherwise there would be no distinction between ' p ⊃ q ' and ' p ≡ q '. And where p implies q without being equivalent with q, this is so because q cannot imply p.

Even when p and q are equivalent we need distinguish between ' p ⊃ q ' and ' q ⊃ p '. For there is a difference between ' p ⊃ p ' and ' p ≡ q '. In the latter case not only are the two propositions p and q different but also ' p ⊃ q ' and ' q ⊃ p ' are different. They are different precisely because their truth conditions are different. Thus FT is a truth condition of ' p ⊃ q ' but falsity condition of ' q ⊃ p '. If therefore this condition is dropped p and q will not only be equivalent but also identical.[6] Thus to say that two propositions are equivalent is not to say that they have identical truth conditions but to say they imply each other, even if these always go hand in hand.

We see therefore that the need for distinguishing between inference of q from p and of p from q necessitates a restriction in the characterization of the inferential relation. And the allegedly paradoxical truth condition of material implications serves just this purpose. Viewed thus we shall not say, as is commonly said, that material implication could pass for the inferential relation but for this paradoxical truth condition of it. We shall rather say that but for this condition the doctrine of material implication could not pass for an adequate doctrine of inferential relation. Thus so-called paradox of material implication is not an avoidable consequence of the fiat of the authors of PM but a necessary component of the notion of material implication considered as identical with the relation of inferribility.[7]

III

We have shown that the paradoxical truth condition, if it be so, of material implication is a necessary part of the meaning of material implication considered as the relation of inferribility. This, however, does not mean, as we hope to show in the next section, that this (paradoxical) truth condition is the condition for there *obtaining* between two propositions, say, p and q, the relation of inferribility. Inclusion of this in a truthfunctional system may be viewed as the device to show what is otherwise expressed by saying that q is not only as a matter of fact true when p is true, but is necessarily so, if inference of q from p is correct.

In the present section we shall show that ' $\supset$ ' does not always represent a relation of inferribility. This may enable us to see that the doctrine of material implication considered as a doctrine of inference is not vitiated by the alleged paradoxes. Besides representing at times inferential relation, ' $\supset$ ' also functions as a sentential connective. Attempts are made to separate formally these two roles of ' $\supset$ ' in Standard Logic by introducing different symbols to symbolize material implication as a truthfunctional connective and material implication as the relation between premiss and conclusion in a correct inference. We shall try here to make a similar distinction, within the limits of the terminology of PM, without overemphasising the difference. For this we consider it more important to clarify the two notions of truthfunctional connective and inferential relation as they stand to each other. These two notions, though distinct, are closely related.[8] This fact rather than the inability of the authors of PM to distinguish the two, is really responsible for to whatever extent the two notions have come to coalesce.[9]

Considered as a truthfunctional sentential connective ' $\supset$ ' is not a primitive notion. It can on the other hand be defined in terms of other notions such as those of negation and disjunction. But considered as representing the relation of inference the notion of material implication is primitive.[10]; at least it is more basic than other truthfunctional connectives. Material implication in this sense cannot be defined in terms of other truthfunctional connectives. For truthfunctional connectives themselves involve, in some indirect sense, the notion of inferribility.[11] For a truth-function is that the truth value of which is uniquely determined by the truth value of its arguments. Thus when we say ' p $\vee$ q ' is a (disjunctive) truthfunction we mean that its truth value can be ascertained if we know the truth value of its compomnent or components. But to say this is to say that the truth value of the compound (truthfunction) can be inferred from the known truth value of its components. Thus when we say ' p $\vee$ q ' is true if p is true we may put the matter as well in this way :

$$\begin{array}{l} p \\ \therefore p \vee q \end{array}$$

But, as we shall see, though the notion of inference is involved in some way in the notion of truthfunction yet this is not the only form of inference or even the most important kind of inference. To distinguish this sort of inference from what we consider the most important sort of inference we shall reserve the term inference for the latter kind of inference only. We shall come to this later.

If in this way the notion of inference is involved in the notion of truthfunction, the notion of inference cannot be reduced to or defined in terms of truthfunction. Or at least there should be different types of inference.[12]

But what holds of '$p \vee q$' or '$p . q$' holds also of '$p \supset q$' since '$p \supset q$' is a truthfunctional compound of p and q. If thus '$p \supset q$' is a truthfunction and hence presupposes the notion of inferential relation, how can it represent the relation of inference itself ? There is, however, a difference. When we treat '$p \supset q$' as a truthfunctional compound we suggest that some definite truth value of '$p \supset q$' is inferrible from some known truth value of p or q or both. Thus '$\sim p \therefore p \supset q$', '$q \therefore p \supset q$', '$p.q \therefore p \supset q$' and so on. None of these meanings of material implication (considered as a sentential connective) allows us to infer q from p. But considered as an inferential relation material implication enables us to do just this. We should therefore, distinguish the truthfunctional derivation of '$p \supset q$' (or '$p \vee q$' etc.) from the derivation of q from p. We may say that truthfunctional derivation of '$p \supset q$' amounts to a construction of an implication in the dissolution of which inference consists.[13]

Truthfunctional derivation of '$p \supset q$' may be said to be adding (compounding) q to (with) p by way of implication just as '$p \vee q$' means adding q to p by way of disjunction. Addition of one proposition to another, i.e., construction of a compound must satisfy certain conditions if the constructed compound is to be a truthfunction of the propositions added. And unless we keep distinct the implication as merely a (truth-functional) sentential connective from implication as the relation of inference, we may take these very conditions to validate the inference of the implicate from the implicans.

But how to formulate and express this difference without using two different symbols to represent the two roles of '$\supset$'? To begin with we may say when '$\supset$' replaces words like 'because', 'therefore', etc., '$\supset$' represents the relation of inference. Thus as a test for whether '$\supset$' represents a relation of inference we are to try to replace '$\supset$' by such words as 'therefore', 'because' etc. In '$\sim p \supset (p \supset q)$' we can replace the first occurrance of '$\supset$' by 'therefore' and rewrite the function as "$\sim p$ therefore '$p \supset q$'." So in its first occurrence here '$\supset$' represents a relation of inference. But '$\supset$' occurring between p and q cannot be regarded as representation of inferential relation since we cannot here replace '$p \supset q$' by 'p therefore q'. Thus though we have here in '$p \supset q$' a truth functional implication we have here no representation of inferential relation. This is why we can say that $\sim p$ inferentially yields '$p \supset q$', but since the latter does not amount to an inference of q from p, we cannot say that the falsity of p entails q. To assert that '$p \supset q$' is true, is not to assert that q is true. Again in '$[(p \supset q) \cdot p] \supset q$' the second occurrence of '$\supset$' is a representation of inferential relation.[14]

Thus in '$\sim p \supset (p \supset q)$' or '$q \supset (p \supset q)$' the second occurrence of '$\supset$' is not a representation of the relation of inference. But in its first occurrence '$\supset$' represents the relation of inference. Likewise the second occurrence of '$\supset$' in $[(p \supset q) . p] \supset q$' is a representation of inferential relation. But even here there is a difference. The first represents the inference of a truthfunctional ('$p \supset q$') from its component $\sim p$' (or, say, q). The second is the representation of the inference of a part of a truthfunctional compound (q) from the truthfunction itself i.e. ($p \supset q$) along with the other part of the compound (p). This second, that is, the derivation of a part of a truthfunctional compound from the compound itself is in the important sense inference. But the derivation of a truth-functional compound from part or parts of it is more a matter of construction rather than inference in the important sense. In a formal system like PM inference or demonstration is primarily understood in this important sense.[15] The proposition to be demonstrated is first shown by substitution etc. to be contained in another proposition, which is already asserted. Then on the basis of the other asserted part of it the proposition under consideration is asserted or inferred.

Thus though a material implication (i.e., an implicative compound) can be derived from or established by, say falsity of the antecedent, yet, when this condition holds no inference in the important sense, is possible; that is, yet there is no inferential relation proper.[16] Only when we add a suitable proposition, i.e., assert a part of the implicative, along with the implicative itself, can there follow the other part. Thus in the important sense inferential relation is represented by the second occurrence of ' $\supset$ ' in such formula as ' $[(p \supset q) . p] \supset q$ '.

Thus we are to distinguish the following three cases of material implication :

(i) ' $p \supset q$ '

(ii) ' $[(p \supset q) . p] \supset q$ '

(iii) ' $\sim p \supset (p \supset q)$ '

The first is not a case of inference at all. For elementary propositions being logically independent of each other cannot stand to each other in inferential relation. We can further assert that no elementary proposition can be inferred from the negation of another elementary proposition. An elementary proposition can be inferred from another elementary proposition only via a truthfunctional compound which contains both the elementary propositions under consideration. Thus (ii) is a legitimate case of inference. The third is also not a case of inference proper for to consider it as a case of inference proper exposes one to the comments similar to those contained in Prior's article Round

About Inference-Ticket. And even if this is viewed as an instance of inference the conclusion is '$p \supset q$' in which none of the two is asserted. Thus $\sim p$ cannot be said to entail q, or, for that matter, a false proposition cannot be said to entail a true proposition.

We can, therefore, finally say that we can take '$\supset$' as representing an inferential relation if it occurs between a proposition (simple or compound) in the place of a conclusion and a conjunctive proposition of which one part is an implicative compound combining the proposition in the conclusion place and the other conjunct. Or we may put this restriction to inferential movement. When there occurs in the implicate position an (implicative) compound we should not allow the consequent of that compound to be asserted.

IV

We have seen that FT condition is a necessary part of the meaning of material implication considered as the relation of inferribility. But when this condition holds it is possible to construct the compound '$p \supset q$' such that the latter is a truthfunctional compound. If this construction of the compound is taken to be a case of inference then also q is not represented as the conclusion of that inference. Thus we concluded that falsity of p did not entail q. In any case FT is not a condition for there obtaining a relation of inference between p and q. The relation of inference obtains between '$(p \supset q) . p$' (i.e., say P) on the one hand and q (i.e., say Q) on the other. And in P $\supset$ Q the '$\supset$' represents a relation of inference proper for here P stands for the conjunction of an implication with one part of that implication for the other part of which Q stands. And though FT is a necessary part of the meaning of '$\supset$' as it occurs in P $\supset$ Q yet it is not a necessary condition for there obtaining between P and Q the relation of inference. It is necessary so that P $\supset$ Q may be distinguished from Q $\supset$ P. Neither is it a sufficient condition for there obtaining between P and Q the relation of inference. It is, however, a sufficient condition for there obtaining a truthfunctional implicative relation between p and q. In other words FT condition is a sufficient condition for the construction of an implicative compound with the true proposition

(viz. q) in the implicate position and the proposition which is false (viz. p) in the implicans position. Paradox is generated when the sufficient condition for construction of a truthfunctional compound is at the same time taken to be the sufficient condition for there obtaining a relation of inference, that is, for the derivation of a part of the compound from the compound itself.

Thus we hold that FT condition is to be viewed as a device to distinguish between the inferences '$P \supset Q$' and '$Q \supset P$'. It is to be further viewed as a device to express a certain feature of the inferential relation which is otherwise expressed by saying that if the relation permitting correct inference obtains between P and Q then Q does not only follow from P as a matter of fact but as a matter of necessity.

This, therefore, is another feature of the intuitive notion of inference or, to be more accurate, intuitive notion of correct inference : derivation of Q from P is intuitively correct if assertion (truth) of P constitutes a sufficient justification for the assertion (truth) of Q.[17] And when is this the case ? When the falsity of Q is a necessary condition for the falsity of P? That is, to say inference of Q from P is a correct inference if P cannot be false unless Q is false. Or falsity of Q is a sufficient condition for the correct inference of falsity of P. Now we can represent the fact that P cannot be true if Q is false, when there obtains a relation permitting the valid deduction of Q from P, if we can show that if P is true then Q is also true; that is to say, if we disallow the combination TF.

But there is another feature of our intuitive notion of correct inference. According to this notion it is not only the case that when P is true Q is also true, but also, that when P is not true Q may or may not be true. We can surely infer the wetness of the ground from there being a rain. But if it does not rain the ground need not necessarily be dry. The conclusion still retains its " hypothetical validity ".[18] Thus intuitively it is clear that if the premiss is false the conclusion may or may not be false. We think, therefore that the intuitive notion of correct inference is restrictive in that it disallows any inferential movement from the falsity of the premiss. Now this feature can be adequately represented in our symbolism if it shows the neutrulity of the falsity of the premiss to the truth or falsity of the conclusion.[19] The doctrine of material implication and the symbolism of PM succeed in doing this. For analysis of the notion of material implication clearly shows that when P is false the relation of correct inference (material implication) holds, if it holds at all, whether Q is true or false. Thus it permits both FT and FF conditions as truth conditions of $P \supset Q$. This shows that we are equally justified, if we are justified at all, to assert falsity of Q under condition of $\sim P$ if we are justified to assert the truth of Q under the same condition. That is if we allow Q to be inferred from $\sim P$ then, since $\sim Q$ may as well be inferred, contradiction will result. This is to be taken as disallowing any inferential move from the falsity of the premiss. This is another reason why FT condition should be retained. This condition along with the condition FF adequately restricts the notion of material implication so that the relation of material implication can pass for the relation of inferribility.

P. K. Mukhopadhyay

University of Jadavpur,
Calcutta.

NOTES

1. Some of the best clarifications of the notion of material implication, within the limits of truth-functional logic, have been suggested by Russell himself. Even before Johnson, for example, Russell had shown the harmlessness of the paradoxes of material implication. See note 16 below.

2. Quine has shown (see his article, *Implication and the Conditional* in *Logic and Philosophy*, ed. Gary Iseminger, Appleton-century crofts) that it is possible to maintain, without abandoning the policy of admitting none but truthfunctional compounds, that certain relations between sentences depend on something other than the truth values. So far we are not opposed to Quine. We only do not wish that intensional concepts be involved in the analysis of material implication considered as the relation of inferribility.

3. Quoted by Von Wrignt in his *Logical Studies* (Routledge and Kegan Paul) p. 166.

4. Robert Ackermann, *An Introduction to Many-Valued Logics* (Routledge and Kegan Paul), p. 6.

5. It will not do to say that even if we drop FT condition '$q \supset p$' can be distinguished from '$p \supset q$' with reference to TF condition which is a falsity condition of '$p \supset q$' but truth condition of '$q \supset p$'. For this is an analogous paradoxical truth condition for '$q \supset p$'. And the question is whether we should at all retain this sort of truth condition as a part of the meaning of '$\supset$'.

6. " ....Two truthfunctions which have the same truth-value for all values of the argument are indistinguishable." Russell, Introduction to Mathematical Philosophy (George Allen & Unwin Ltd.), p. 148.

7. It may appear that we suggest that the joint truth and falsehood of two propositions are sufficient conditions for there obtaining, between the two propositions, the relation of inference. In a sense this is true. But we insist on a certain order in the truth value distribution. This order cannot immediately be represented in the symbolism of PM. But we can utilize FT condition to make good this deficiency to a large extent. (See below note 17). So we do not think that the charges of Anderson and Belnap affect the present interpretation of material implication. Anderson and Belnap, *Entailment in Logic and Philosophy* ed. Iseminger (Appleton-Century-Crafts), p. 78.

8. '$p \supset q$' is not the same as 'q is inferrible from p'. But whenever there obtains [e.g. between '$(p \supset q) . p$' and 'q'] the relation of inference there also obtains a material conditional. The converse need not and perhaps cannot be read into PM.

9. Russell, *Introduction to Mathematical Philosophy*, p. 151.

10. Russell seems to have seen this point but he did not elaborate it. Russell, *ibid.*, p. 146.

11. Russell, *ibid.*, p. 146.

12. Russell, *ibid.*, p. 151.

13. Whitehead and Russell, *Principia Mathematica* (referred in the text as PM) (Cambridge University Press, Paperback), p. 9.

14. " ....'$\vdash p \supset \vdash q$'....is to be considered as a mere abbreviation of the threefold statement '$\vdash p$' and '$\vdash (p \supset q)$' and '$\vdash q$'. Thus '$\vdash p \supset \vdash q$' may be read '$p$, therefore q'. Russell, *ibid.*, p. 9.

15. " The process of inference is as follows : a proposition 'p' is asserted, and a proposition 'p implies q' is asserted, and then as a sequel the proposition 'q' is asserted." Russell, *ibid.*, pp. 8–9.

" In mathematical practice, when we infer, we have always some expression containing variable propositions, say p and q, which is known, in virtue of its form, to be true for all values of p and q; we have also some other expression, part of the former, which is also known to be true for all values of p and q; and in virtue of the principles of inference; we are able to drop this part of our original expression, and assert what is left." Russell, *Introduction to Mathematical Philosophy*, p. 149.

16. " Whenever p is false, ' not—p or q ' is true, but is useless for inference, which requires that p should be true." Russell, *ibid*, p. 153. also *Principia Mathematica*, p. 94.

17. " The essential property that we require of implication is this : 'What is implied by a true proposition is true '." Russell, *ibid*, p. 94. Russell also says that the falsity of p does not yield q. This two together constitute adequate restrictive condition of material implication. But the difficulty should not be minimised. These two along with the fact that TF is a falsity condition merely tell us that truth-value distribution between the implication and the implicans should be uniform. That is, either both are true or both are false. But what is necessary is that when p is true q must be true and when q is false p must be false. This order of truth-value distribution, however, cannot satisfactorily be expressed in the symbolism oI PM. For '$(p \cdot q) . (\sim q . \sim p)$' has no meaning in PM other than $(q.p.) . (\sim p. \sim q)$'. Admitting this limitation we have tried in the paragraph concerned the best that can be done in the informal way. But at the same time we maintain that sufficient indication towards this interpretation is there is Russell's own writings.

18. Russell, *Introduction to Mathematical Philosophy*, p. 198.

19. Though Russell does not tell us explicitly whether he means this neutrality is already represented by the truth table or by any other means yet he explicitly says that this neutrality is there. " The essential property that we require of implication is this : ' What is implied by a true proposition is true ', It is in virtue of this property that implication yields proofs. But this property by no means determines whether anything, and if so what, is implied by false proposition." Russell, *Principia Mathematica*, p. 94.

* Since the discussion in this paper is informal I have avoided quotation marks etc. unless there is serious chance of misunderstanding.